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# EXPLICIT FORMULAS FOR WHITTALKER FUNCTIONS ON CLASSICAL GROUPS

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**Abstract:** We give explicit recursive relations for Whittaker functions on classical groups and its application to archimedean zeta integrals.

**Keywords:** Whittaker functions, automorphic $L$-functions.

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## Introduction

In the theory of automorphic forms Whittaker functions play very important role. In this note we collect our recent results ([4], [2], [3]) on explicit formulas for Whittaker functions on classical groups.

Let us briefly recall the simplest case $SL_2(\mathbb{R})$. A Maass wave form $f$ is an automorphic form on the upper half plane $\mathfrak{H} = \{z = x + \sqrt{-1}y \mid y > 0\}$, which is an eigenfunction of the Laplacian of $\mathfrak{H}$, that is, $f$ satisfies

$$-y^2\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big)f(z) = \Big(\frac{1}{4} - \nu^2\Big)f(z)$$

($\nu \in \mathbb{C}$). Then $f$ has the Fourier expansion of the form

$$\begin{aligned} f(x + \sqrt{-1}y) &= ay^{\nu+1/2} + by^{-\nu+1/2} \\ &+ \sum_{n\neq 0} a_n\sqrt{y}K_\nu(2\pi|n|y)\exp(2\pi\sqrt{-1}nx), \end{aligned}$$

where $K_\nu(z)$ is the modified $K$-Bessel function (= *class one Whittaker function* on $SL_2(\mathbb{R})$) and satisfies Bessel's differential equation

$$\Big[\Big(z\frac{d}{dz}\Big)^2 - (z^2 + \nu^2)\Big]K_\nu(z) = 0.$$

When $\nu \notin \mathbb{Z}$, the fundamental solution of this differential equation around $z = 0$ is $\{I_\nu(z), I_{-\nu}(z)\}$ with

$$I_\nu(z) = \sum_{m=0}^{\infty} \frac{(z/2)^{2m+\nu}}{m!\,\Gamma(\nu + m + 1)}$$

the $I$-Bessel function (= *fundamental Whittaker function* on $SL_2(\mathbb{R})$) and there is the relation

$$K_\nu(z) = \frac{\pi}{2\sin\nu\pi}\big(I_{-\nu}(z) - I_\nu(z)\big).$$

We shall discuss explicit formulas of these special functions on higher rank groups. Our explicit formulas are applicable for archimedean theory of automorphic $L$-functions such as the computation of archimedean $L$-factors.

## 1. Whittaker functions for class one principal series representations

We recall the notion of Whittaker functions for class one principal series representations of real semisimple Lie groups. Let $G$ be a real semisimple Lie group with finite center and $\mathfrak{g}$ its Lie algebra. Fix a maximal compact subgroup $K$ of $G$ and put $\mathfrak{k} = \mathrm{Lie}(K)$. Let $\mathfrak{p}$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ and $\theta$ the corresponding Cartan involution. For a maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$ and $\alpha \in \mathfrak{a}^*$, put $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid [H, X] = \alpha(H)X, \forall H \in \mathfrak{a}\}$ and $\Delta = \Delta(\mathfrak{g}, \mathfrak{a})$ the restricted root system. Denoted by $\Delta^+$ the positive system in $\Delta$ and $\Pi$ the set of simple roots. Then we have an Iwasawa decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{k}$ with $\mathfrak{n} = \sum_{\alpha\in\Delta^+} \mathfrak{g}_\alpha$. Let $G = NAK$ be the Iwasawa decomposition corresponding to that of $\mathfrak{g}$. We denote by $\mathcal{W}$ the Weyl group of the root system $\Delta$.

Let $P = MAN$ be a minimal parabolic subgroup of $G$ with $M = Z_K(A)$. For a linear form $\nu \in \mathfrak{a}^*_\mathbb{C} = \mathfrak{a}^* \otimes_\mathbb{R} \mathbb{C}$, define a character $\exp(\nu)$ on $A$ by $\exp(\nu)(a) = \exp(\nu(\log a))$ ($a \in A$). We call the induced representation

$$I(\nu) = L^2\text{-}\mathrm{Ind}_P^G(1_M \otimes \exp(\nu + \rho) \otimes 1_N)$$

the *class one principal series representation* of $G$. Here $\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+} m_\alpha\alpha$ is the half sum of positive roots ($m_\alpha = \dim \mathfrak{g}_\alpha$).

Let $\eta$ be a unitary character of $N$. Since $\mathfrak{n} = [\mathfrak{n}, \mathfrak{n}] \oplus \sum_{\alpha\in\Pi} \mathfrak{g}_\alpha$, $\eta$ is determined by the restriction $\eta_\alpha := d\eta|_{\mathfrak{g}_\alpha}$ ($\alpha \in \Pi$). The length $|\eta_\alpha|$ of $\eta_\alpha$ is defined as $|\eta_\alpha|^2 = \sum_{1\le i\le m(\alpha)} -d\eta(X_{\alpha,i})^2$ (note that $d\eta(X_{\alpha,i}) \in \sqrt{-1}\mathbb{R}$), where the root vector $X_{\alpha,i}$ is chosen as $B(X_{\alpha,i}, \theta X_{\alpha,j}) = -\delta_{i,j}$ ($1 \le i, j \le m(\alpha)$). Here $B(\ ,\ )$ is the Killing form on $\mathfrak{g}$. In this note we assume that $\eta$ is nondegenerate, that is, $\eta_\alpha \neq 0$ for all $\alpha \in \Pi$. For such $\eta$, the space

$$\begin{aligned} C^\infty_\eta(N\backslash G) = \{&f \in C^\infty(G, \mathbb{C}) \\ &\mid f(ng) = \eta(n)f(g), \forall(n, g) \in N \times G\} \end{aligned}$$

becomes a $(\mathfrak{g}_\mathbb{C}, K)$-module. For a nonzero intertwining operator $\Phi$ between $I(\nu)$ and $C^\infty_\eta(N\backslash G)$

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and a spherical vector $v_0 \in \pi_\nu$, we call the image $\Phi(v_0)$ the *Whittaker function for the class one principal series*. We denote by

$$\begin{aligned}&\mathrm{Wh}(\nu,\eta)\\&=\{\Phi(v_0) \mid \Phi \in \mathrm{Hom}_{(\mathfrak{g}_\mathbb{C},K)}(I(\nu), C_\eta^\infty(N\backslash G))\}\end{aligned}$$

the space of Whittaker functions. Because of Iwasawa decomposition, Whittaker function $w$ is determined by its restriction $w|_A$ to $A$, which we call the *radial part* of $w$. Let $\mathrm{Wh}(\nu,\eta)^{\mathrm{mod}}$ be the subspace of $\mathrm{Wh}(\nu,\eta)$ consisting of moderate growth functions. Here is the well-known results (see [7], [9]):

- $\dim_\mathbb{C} \mathrm{Wh}(\nu,\eta) = |\mathcal{W}|$,
- $\dim_\mathbb{C} \mathrm{Wh}(\nu,\eta)^{\mathrm{mod}} = 1$.

Hashizume [1] constructed a basis of the space $\mathrm{Wh}(\nu,\eta)$ by a power series $M_{\nu,\eta}$ whose coefficients are characterized by a recurrence relation coming from Casimir operators. In the next section we review the construction of $M_{\nu,\eta}$, which we call *fundamental Whittaker functions*.

The unique moderate growth Whittaker function has an integral representation introduced by Jacquet [6]:

$$W_{\nu,\eta}(g) = \int_N \eta^{-1}(n)a(w^{-1}ng)^{\nu+\rho}\,dn$$

where $w$ is the longest element in $\mathcal{W}$ and $g = n(g)a(g)k(g)$ means the Iwasawa decomposition of $g \in G$. It gives a unique moderate growth Whittaker function and it is known that as a function of $\nu$, $W_{\nu,\eta}$ converges absolutely and uniformly on $\{\nu \in \mathfrak{a}_\mathbb{C}^* \mid \mathrm{Re}(\langle \nu,\alpha\rangle) > 0 \text{ for all } \alpha \in \Delta^+\}$ and can be continued to a meromorphic function. We refer the Jacquet integral *class one Whittaker function*. Following the idea of Harish-Chandra, Hashizume expressed the Jacquet integral as a linear combination of the fundamental Whittaker functions. This *factorization formula* plays important role to obtain an integral representation of class one Whittaker function.

In this paper we discuss the case of classical group $G = SL_{n+1}(\mathbb{R})$, $SO_{n+1,n}(\mathbb{R})$, $Sp_n(\mathbb{R})$ and $SO_{n,n}(\mathbb{R})$. Here $SO_{n+1,n}(\mathbb{R})$ (resp. $SO_{n,n}(\mathbb{R})$) is the special orthogonal group with respect to $J_{2n+1}$ (resp. $J_{2n}$), and $Sp_n(\mathbb{R})$ is the symplectic group with respect to $\begin{pmatrix} & J_n \\ -J_n & \end{pmatrix}$ with $J_n = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}$ ($n \times n$ matrix). To describe our explicit formulas for Whittaker functions we fix some notation for coordinates on $A$ and parameters for principal series. In our realization of $G$, $a \in A$ is of the form

$$a = \begin{cases} \mathrm{diag}(a_1,\dots,a_{n+1}) \\ \qquad \text{if } G = SL_{n+1}(\mathbb{R}); \\ \mathrm{diag}(a_1,\dots,a_n,1,a_n^{-1},\dots,a_1^{-1}) \\ \qquad \text{if } G = SO_{n+1,n}(\mathbb{R}); \\ \mathrm{diag}(a_1,\dots,a_n,a_n^{-1},\dots,a_1^{-1}) \\ \qquad \text{if } G = Sp_n(\mathbb{R}), SO_{n,n}(\mathbb{R}), \end{cases}$$

with $a_i > 0$. We introduce new coordinates $y = (y_1,\dots,y_n)$ on $A$ by $y_i = a_i/a_{i+1}$ $(1 \le i \le n-1)$ and

$$y_n = \begin{cases} a_n/a_{n+1} & \text{if } G = SL_{n+1}(\mathbb{R}); \\ a_n & \text{if } G = SO_{n+1,n}(\mathbb{R}); \\ a_n^2 & \text{if } G = Sp_n(\mathbb{R}); \\ a_{n-1}a_n & \text{if } G = SO_{n,n}(\mathbb{R}). \end{cases}$$

The parameter $\nu$ of principal series of $G$ is identified with the complex numbers $(\nu_1,\dots,\nu_n)$ by

$$\exp(\nu)(a) = \begin{cases} \prod_{i=1}^{n+1} a_i^{2\nu_i} & \text{if } G = SL_{n+1}(\mathbb{R}); \\ \prod_{i=1}^{n} a_i^{2\nu_i} & \text{otherwise.} \end{cases}$$

Note that $\sum_{i=1}^{n+1} \nu_i = 0$ for $G = SL_{n+1}(\mathbb{R})$.

## 2. Fundamental Whittaker functions

In this section we give explicit formulas for fundamental Whittaker functions. We first review Hashizume's result. Let $\langle\ ,\ \rangle$ be the inner product on $\mathfrak{a}_\mathbb{C}^*$ induced by the Killing form $B(\ ,\ )$. We denote by $L$ the set of linear functions on $\mathfrak{a}_\mathbb{C}$ of the form $\sum_{\alpha\in\Pi} n_\alpha \alpha$ with $n_\alpha \in \mathbb{Z}_{\ge 0}$.

For each $\lambda \in L$, we can define the rational function $c_\lambda(\nu)$ on $\mathfrak{a}_\mathbb{C}^*$ as follows. Put $c_0(\nu) = 1$ and determine $c_\lambda(\nu)$ for $\lambda \in L\backslash\{0\}$ by

$$\begin{aligned}&(\langle\lambda,\lambda\rangle + 2\langle\lambda,\nu\rangle)c_\lambda(\nu)\\&= 2\textstyle\sum_{\alpha\in\Pi} |\eta_\alpha|^2 c_{\lambda-2\alpha}(\nu),\end{aligned} \tag{2.1}$$

inductively. Here we assumed $\langle\lambda,\lambda\rangle + 2\langle\lambda,\nu\rangle \ne 0$ for all $\lambda \in L\backslash\{0\}$.

We define a series $M_{\nu,\eta}(a)$ on $A$ by

$$M_{\nu,\eta}(a) = a^{\nu+\rho}\sum_{\lambda\in L} c_\lambda(\nu)a^\lambda \quad a \in A,$$

and ($a^{\nu+\rho} = \exp(\nu+\rho)(a)$) extend it to the function on $G$ by

$$M_{\nu,\eta}(g) = \eta(n(g))M_{\nu,\eta}(a(g)).$$

We call the power series $M_{\nu,\eta}$ the *fundamental Whittaker function* on $G$. Hashizume proved the following.

**Proposition 1.** ([1, Theorem 5.4]) *If $\nu \in \mathfrak{a}_\mathbb{C}^*$ is regular, then the set*

$$\{M_{s\nu,\eta}(g) \mid s \in \mathcal{W}\}$$

*forms a basis of* $\mathrm{Wh}(\nu,\eta)$*. Here an element $\nu \in \mathfrak{a}_\mathbb{C}^*$ is called regular if the following two conditions are satisfied.*

- $\langle \lambda, \lambda \rangle + 2\langle \lambda, s\nu \rangle \neq 0$ *for all* $\lambda \in L \backslash \{0\}$ *and* $s \in \mathcal{W}$,
- $s\nu - t\nu \notin \{\sum_{\alpha \in \Pi} m_\alpha \alpha \mid m_\alpha \in \mathbb{Z}\}$ *for all* $s \neq t \in \mathcal{W}$.

Therefore to find explicit formulas of Whittaker functions, we need to solve the recurrence relation (2.1). If we suitably fix the parameters $\eta_\alpha$ and use coordinates $y = (y_1, \ldots, y_n)$ on $A$, then the radial parts of fundamental Whittaker functions can be written as of the form:

$$M_\nu^G(y) = y^{\rho+\nu} \sum_{\mathbf{m}=(m_1,\ldots,m_n)\in\mathbb{N}^n} C_{\mathbf{m}}^G(\nu) \times (\pi y_1)^{2m_1} \cdots (\pi y_n)^{2m_n}.$$

Here the coefficients $C_{\mathbf{m}}^G(\nu)$ are characterized by the initial condition $C_{(0,\ldots,0)}^G(\nu) = 1$ and the recurrence relation:

$$Q_{\mathbf{m}}^G(\nu)\, C_{\mathbf{m}}^G(\nu) = \sum_{i=1}^{n-1} C_{\mathbf{m}-\mathbf{e}_i}^G(\nu) + \varepsilon^G\, C_{\mathbf{m}-\mathbf{e}_n}^G(\nu),$$

where we denote by $\mathbf{e}_i$ the $i$-th standard basis in $\mathbb{R}^n$ and $Q_{\mathbf{m}}(\nu)$ and $\varepsilon^G$ are given as follows:

$$\begin{aligned} Q_{\mathbf{m}}^G(\nu) &= \textstyle\sum_{i=1}^{n-1} m_i^2 + \varepsilon^G m_n^2 \\ &\quad - \textstyle\sum_{i=1}^{n-2} m_i m_{i+1} + \sum_{i=1}^{n-1}(\nu_i - \nu_{i+1}) m_i \\ &\quad + \begin{cases} -m_{n-1}m_n + (\nu_n - \nu_{n+1})m_n \\ \qquad \text{if } G = SL_{n+1}(\mathbb{R}); \\ -m_{n-1}m_n + \nu_n m_n \\ \qquad \text{if } G = SO_{n+1,n}(\mathbb{R}); \\ -2m_{n-1}m_n + 2\nu_n m_n \\ \qquad \text{if } G = Sp_n(\mathbb{R}); \\ -m_{n-2}m_n + (\nu_{n-1} + \nu_n)m_n \\ \qquad \text{if } G = SO_{n,n}(\mathbb{R}), \end{cases} \end{aligned}$$

and

$$\varepsilon^G = \begin{cases} 1 & \text{if } G = SL_{n+1}(\mathbb{R}), SO_{n,n}(\mathbb{R}); \\ 1/2 & \text{if } G = SO_{n+1,n}(\mathbb{R}); \\ 2 & \text{if } G = Sp_n(\mathbb{R}). \end{cases}$$

We solve the recurrence relations above and find recursive formulas with respect to the real rank of $G$. For $G = SL_{n+1}(\mathbb{R})$, $SO_{n+1,n}(\mathbb{R})$, $Sp_n(\mathbb{R})$ and $SO_{n,n}(\mathbb{R})$, we write $G' = SL_n(\mathbb{R})$, $SO_{n,n-1}(\mathbb{R})$, $Sp_{n-1}(\mathbb{R})$ and $SO_{n-1,n-1}(\mathbb{R})$, respectively. For the class one principal series $I(\nu)$ of $G$, we associate principal series $I(\tilde{\nu})$ of $G'$ with $\tilde{\nu} = (\tilde{\nu}_1, \ldots, \tilde{\nu}_{n-1})$ as follows.

$$\tilde{\nu}_i = \begin{cases} \nu_{i+1} + \nu_1/n & \text{if } G = SL_{n+1}(\mathbb{R}), \\ \nu_i & \text{otherwise.} \end{cases}$$

Then we obtained the following.

**Theorem 2.** ([4], [2], [3]) *The coefficients* $C_{\mathbf{m}}^G(\nu)$ *can be written in terms of* $C_{\mathbf{k}=(k_1,\ldots,k_{n-1})}^{G'}(\tilde{\nu})$ *:*

$$C_{\mathbf{m}}^G(\nu) = \sum_{\mathbf{k}\le\mathbf{m}} \frac{C_{\mathbf{k}}^{G'}(\tilde{\nu})}{P_{\mathbf{m},\mathbf{k}}^G},$$

*where*

$$\begin{aligned} &P_{\mathbf{m},\mathbf{k}}^{SL_{n+1}(\mathbb{R})} \\ &\quad = \textstyle\prod_{i=1}^n (m_i - k_i)!(\nu_i - \nu_{n+1} + 1)_{m_i - k_{i-1}}, \end{aligned}$$

$$\begin{aligned} &P_{\mathbf{m},\mathbf{k}}^{SO_{n+1,n}(\mathbb{R})} \\ &\quad = \frac{\prod_{i=1}^{n-1}(m_i - k_i)!(\nu_i + \nu_{i+1} + 1)_{m_{i+1} - k_{i-1}}}{\prod_{i=1}^{n-1}(\nu_i + \nu_{i+1} + 1)_{m_i + m_{i+1} - k_{i-1} - k_i}} \\ &\quad \times \textstyle\prod_{i=1}^{n-1}(\nu_i - \nu_n + 1)_{m_i - k_{i-1}} \\ &\quad \times \textstyle\prod_{i=1}^{n-1}(\nu_i + \nu_n + 1)_{m_i - k_{i-1}} \\ &\quad \times (m_n - k_{n-1})!(2\nu_n + 1)_{m_n - k_{n-1}}, \end{aligned}$$

$$\begin{aligned} &P_{\mathbf{m},\mathbf{k}}^{Sp_n(\mathbb{R})} \\ &\quad = \frac{\prod_{i=1}^{n-2}(m_i - k_i)!(\nu_i + \nu_{i+1} + 1)_{m_{i+1} - k_{i-1}}}{\prod_{i=1}^{n-2}(\nu_i + \nu_{i+1} + 1)_{m_i + m_{i+1} - k_{i-1} - k_i}} \\ &\quad \times \textstyle\prod_{i=1}^{n-1}(\nu_i - \nu_n + 1)_{m_i - k_{i-1}} \\ &\quad \times \textstyle\prod_{i=1}^{n-1}(\nu_i + \nu_n + 1)_{m_i - k_{i-1}} \\ &\quad \times \frac{(m_{n-1} - 2k_{n-1})!(m_n - k_{n-1})!}{(\nu_{n-1} + \nu_n + 1)_{m_{n-1} + 2m_n - 2k_{n-1} - k_{n-2}}} \\ &\quad \times (\nu_{n-1} + \nu_n + 1)_{2m_n - k_{n-2}}(\nu_n + 1)_{m_n - k_{n-1}}, \end{aligned}$$

*and*

$$\begin{aligned} &P_{\mathbf{m},\mathbf{k}}^{SO_{n,n}(\mathbb{R})} \\ &\quad = \frac{\prod_{i=1}^{n-3}(m_i - k_i)!(\nu_i + \nu_{i+1} + 1)_{m_{i+1} - k_{i-1}}}{\prod_{i=1}^{n-3}(\nu_i + \nu_{i+1} + 1)_{m_i + m_{i+1} - k_{i-1} - k_i}} \\ &\quad \times \textstyle\prod_{i=1}^{n-2}(\nu_i - \nu_n + 1)_{m_i - k_{i-1}} \\ &\quad \times \textstyle\prod_{i=1}^{n-2}(\nu_i + \nu_n + 1)_{m_i - k_{i-1}} \\ &\quad \times (m_{n-2} - k_{n-2} - k_{n-1})! \\ &\quad \times \frac{(\nu_{n-2} + \nu_{n-1} + 1)_{m_{n-1} + m_n - k_{n-3}}}{(\nu_{n-2} + \nu_{n-1} + 1)_{{}^{n}_{i=n-2} m_i - {}^{n-1}_{i=n-3} k_i}} \\ &\quad \times (m_{n-1} - k_{n-1})!(\nu_{n-1} - \nu_n + 1)_{m_{n-1} - k_{n-2}} \\ &\quad \times (m_n - k_{n-1})!(\nu_{n-1} + \nu_n + 1)_{m_n - k_{n-2}} \end{aligned}$$

*with* $(a)_n = \Gamma(a+n)/\Gamma(a)$, *and* $\mathbf{k}$ *runs through such that* $P_{\mathbf{m},\mathbf{k}}^G$ *do not vanish (finite sum).*

## 3. Class one Whittaker functions

In this section we give integral representations for class one Whittaker functions. Jacquet integral of course is an integral representation for class one Whittaker function, however, it is not satisfactory form for application such as computation of archimedean $L$-factors of automorphic $L$-functions. After long calculation Stade [8] evaluated Jacquet integral on $SL_n(\mathbb{R})$ to reach an effective recursive formula between $SL_n(\mathbb{R})$ and $SL_{n-2}(\mathbb{R})$. Further he computed $L$-factors of $L$-functions on $GL_n \times GL_n$ and $GL_n \times GL_{n-1}$. Based on Stade's result, he and the author [4] found a recursive formula between $SL_n(\mathbb{R})$ and $SL_{n-1}(\mathbb{R})$. In [2], [3], we obtain similar formulas for other classical groups, and our approach here does not based on Jacquet integrals. We use a linear relation between fundamental Whittaker functions and class one Whittaker functions given in [1]. If we set

$$W^G = \sum_{s \in \mathcal{W}} \Gamma^G_{s\nu} \cdot M_{s\nu},$$

then, up to constant (depending on $\nu$), $W^G$ is class one Whittaker function on $G$. Here

$$\begin{aligned}
\Gamma_\nu^{SL_n(\mathbb{R})} &= \textstyle\prod_{1 \le i < j \le n+1} \Gamma(-\nu_i + \nu_j);\\
\Gamma_\nu^{SO_{n+1,n}(\mathbb{R})} &= \Gamma_\nu^{Sp_n(\mathbb{R})}\\
&= \textstyle\prod_{1 \le i < j \le n} \Gamma(-\nu_i + \nu_j)\Gamma(-\nu_i - \nu_j)\\
&\quad \times \textstyle\prod_{1 \le i \le n} \Gamma(-2\nu_i);\\
\Gamma_\nu^{SO_{n,n}(\mathbb{R})} &= \textstyle\prod_{1 \le i < j \le n} \Gamma(-\nu_i + \nu_j)\Gamma(-\nu_i - \nu_j).
\end{aligned}$$

As in Theorem 2 we give recursive relations between class one Whittaker functions on $G$ and $G'$.

**Theorem 3.** ([4], [2], [3]) *If we denote by $W^G(y) = y^\rho \widetilde{W}^G(y)$. Then we have following recursive relations:*

$$\begin{aligned}
&\widetilde{W}_\nu^{SL_{n+1}(\mathbb{R})}(y)\\
&= \int_{(\mathbb{R}_+)^n} \prod_{i=1}^n \exp\Big\{-u_i - \frac{(\pi y_i)^2}{u_i}\Big\} \cdot \widetilde{W}_{\tilde\nu}^{SL_n(\mathbb{R})}(\tilde y)\\
&\quad \times \Big(\prod_{i=1}^n (\pi y_i)^{2(n+1-i)} u_i^{n+1}\Big)^{\frac{\nu_1}{n}} \prod_{i=1}^n \frac{du_i}{u_i};
\end{aligned}$$

$$\begin{aligned}
&\widetilde{W}_\nu^{SO_{n+1,n}(\mathbb{R})}(y)\\
&= \int_{(\mathbb{R}^+)^{n-1}} \prod_{i=1}^{n-1} K_{2\nu_n}\Big(2\pi y_i \sqrt{(1+u_{i-1})(1+\tfrac{1}{u_i})}\Big)\\
&\times K_{2\nu_n}\big(2\pi y_n \sqrt{1+u_{n-1}}\big) \widetilde{W}_{\tilde\nu}^{SO_{n,n-1}(\mathbb{R})}(\tilde y) \prod_{i=1}^{n-1} \frac{du_i}{u_i};
\end{aligned}$$

$$\begin{aligned}
&\widetilde{W}_\nu^{Sp_n(\mathbb{R})}(y)\\
&= \int_{(\mathbb{R}^+)^{n-1}} \prod_{i=1}^{n-1} K_{2\nu_n}\Big(2\pi y_i \sqrt{(1+u_{i-1})(1+\tfrac{1}{u_i})}\Big)\\
&\times K_{\nu_n}\big(2\pi y_n (1+u_{n-1})\big) \widetilde{W}_{\tilde\nu}^{Sp_{n-1}(\mathbb{R})}(\tilde y) \prod_{i=1}^{n-1} \frac{du_i}{u_i};
\end{aligned}$$

$$\begin{aligned}
&\widetilde{W}_\nu^{SO_{n,n}(\mathbb{R})}(y)\\
&= \int_{(\mathbb{R}^+)^{n-1}} \prod_{i=1}^{n-2} K_{2\nu_n}\Big(2\pi y_i \sqrt{(1+u_{i-1})(1+\tfrac{1}{u_i})}\Big)\\
&\times K_{2\nu_n}\Big(2\pi \sqrt{(1+u_{n-2})(1+u_{n-1})(\tfrac{y_{n-1}^2}{u_{n-1}} + y_n^2)}\Big)\\
&\times \widetilde{W}_{\tilde\nu}^{SO_{n-1,n-1}(\mathbb{R})}(\tilde y)\\
&\times \Big(\frac{y_{n-1} y_n (1+u_{n-1})}{y_{n-1}^2 + u_{n-1} y_n^2}\Big)^{\nu_n} \prod_{i=1}^{n-1} \frac{du_i}{u_i},
\end{aligned}$$

*where $\tilde y = (\tilde y_1, \ldots, \tilde y_{n-1})$ with $\tilde y_i = y_{i+1}\sqrt{u_i/u_{i+1}}$ $(1 \le i \le n-2)$ and*

$$\tilde y_{n-1} = \begin{cases} y_n \sqrt{u_{n-1}/u_n} & \text{if } G = SL_{n+1}(\mathbb{R});\\ y_n \sqrt{u_{n-1}} & \text{if } G = SO_{n+1,n}(\mathbb{R});\\ y_n u_{n-1} & \text{if } G = Sp_n(\mathbb{R});\\ y_n \sqrt{u_{n-2} u_{n-1}} & \text{if } G = SO_{n,n}(\mathbb{R}). \end{cases}$$

Finally we announce our recent result on a computation of archimedean $L$-factors.

**Theorem 4.** ([5]) *We have*

$$\begin{aligned}
&\int_{(\mathbb{R}^\times)^n} \widetilde{W}^{SO_{n+1,n}(\mathbb{R})}_{(\nu_1,\ldots,\nu_n)}(y_1, \ldots, y_n)\\
&\times \widetilde{W}^{SL_n(\mathbb{R})}_{(\mu_1,\ldots,\mu_n)}(y_1, \ldots, y_{n-1}) \prod_{i=1}^n y_i^{is} \frac{dy_i}{y_i}\\
&= \frac{\prod_{i=1}^n \prod_{j=1}^n \Gamma_{\mathbb{R}}(s + \nu_i + \mu_j)\Gamma_{\mathbb{R}}(s - \nu_i + \mu_j)}{\prod_{1 \le i < j \le n} \Gamma_{\mathbb{R}}(2s + \mu_i + \mu_j)},
\end{aligned}$$

*and*

$$\begin{aligned}
&\int_{(\mathbb{R}^\times)^n} \widetilde{W}^{SO_{n+1,n}(\mathbb{R})}_{(\nu_1,\ldots,\nu_n)}(y_1, \ldots, y_n)\\
&\times \widetilde{W}^{SL_{n+1}(\mathbb{R})}_{(\mu_1,\ldots,\mu_{n+1})}(y_1, \ldots, y_n) \prod_{i=1}^n y_i^{is} \frac{dy_i}{y_i}\\
&= \frac{\prod_{i=1}^n \prod_{j=1}^{n+1} \Gamma_{\mathbb{R}}(s + \nu_i + \mu_j)\Gamma_{\mathbb{R}}(2s - \nu_i + \mu_j)}{\prod_{1 \le i < j \le n+1} \Gamma_{\mathbb{R}}(2s + \mu_i + \mu_j)}.
\end{aligned}$$

## References


[1] M. Hashizume, Whittaker functions on semisimple Lie groups, Hiroshima Math. J. **12** (1982), 259–293.

[2] T. Ishii, Whittaker functions on orthogonal groups of odd degree, submitted.

[3] T. Ishii, Symplectic orthogonal theta lifts and Whittaker functions, in preparation.

[4] T. Ishii and E. Stade, New formulas for Whittaker functions on $GL(n, \mathbf{R})$, J. Funct. Anal. **244** (2007), 289–314.

[5] T. Ishii and E. Stade, Archimedean $L$-factors on $GL_n \times GL_m$ and $SO_{2n+1} \times GL_m$, in preparation.

[6] H. Jacquet, Fonctions de Whittaker associées aux groupes de Chevalley, Bull. Soc. Math. France **95** (1967), 243–309.

[7] J. Shalika, The multiplicity one theorem for $GL(n)$, Annals of Math **100** (1974), 171–193.

[8] E. Stade, On explicit integral formulas for $GL(n, \mathbf{R})$-Whittaker functions. Duke Math. J. **60** (1990), no. 2, 313–362.

[9] N. Wallach, Asymptotic expansions of generalized matrix entries of representations of real reductive groups, Lect. Notes in Math. **1024** (1984), 287–369.