成蹊大学理工学研究報告
J. Fac. Sci.Tech., Seikei Univ.
Vol.42 No.1 (2005)pp.57-62

On the exponential equations $a^x - b^y = c\,(1 \le c \le 300)$

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Abstract

M. A. Bennett obtained the following theorem in 2001. If a, b, c are positive integers with $a, b \ge 2$, and $1 \le c \le 100$, then the equation $a^x - b^y = c$ has at most one solution in positive integers x and y except the ten exceptional cases. In this paper we will study the same problem in the case of $1 \le c \le 300$. We follow Bennett's method and we use a slight idea in the place where we use computers.

Keywords: Exponential equations, Baker's methods, Continued fraction expansion

(Received March 25, 2005)

1 Introduction

M. A. Bennett studied the Diophantine equation

$$a^x - b^y = c \tag{1}$$

where a, b, c positive integers with $a, b \ge 2$, and $1 \le c \le 100$. From now on, a triple (a, b, c) means the Diophantine equation (1). He obtained the following result([1], Theorem 1.5).

The equation (1) has at most one solution in positive integers x and y, except for triples (a, b, c) satisfying $(a, b, c) \in \{(3, 2, 1), (2, 3, 5), (2, 3, 13), (4, 3, 13), (16, 3, 13), (2, 5, 3), (13, 3, 10), (91, 2, 89), (6, 2, 4), (15, 6, 9)\}$. In each of these cases, (1) has precisely two positive solutions.

We will try to extend the above result to the cases of $1 \le c \le 300$.

We obtain

Theorem *If a, b, c are positive integers with $a, b \ge 2$, and $1 \le c \le 300$, then the equation (1) has at most one solution in positive integers x and y, except for a triple (a, b, c) satisfying $(a, b, c) = (280, 5, 275)$ besides the above ten cases. In this exceptional case, (1) has precisely two positive solutions.*

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2 Necessary Results

Before we proceed with the proof of our Theorem, we will mention a related result due to Scott([5]).

Proposition 1 *$b > 1$ and c are positive integers and a is a positive rational prime, then equation (1) has at most one solution in positive integers x and y unless either $(a, b, c) = (3, 2, 1), (2, 3, 5), (2, 3, 13)$ or $(2, 5, 3)$, or $a > 2$, $\gcd(a, b) = 1$ and the smallest $t \in \mathbf{N}$ such that $b^t \equiv 1 \pmod{a}$ satisfies $t \equiv 1 \pmod{2}$. In these situations, the given equation has at most two solutions. If equation (1), with the above hypotheses, has distinct solutions (x_1, y_1) and (x_2, y_2), then $y_2 - y_1 \equiv 1 \pmod{2}$, unless $(a, b, c) = (3, 2, 1), (2, 3, 5), (2, 3, 13), (2, 5, 3)$, or $(13, 3, 10)$*

As Bennett pointed out at the end of §2 in [1], we may assume $a \ge 6$ and we will henceforth assume, without loss of generality, that a and b are not perfect powers.

The following is the corollary to Theorem 2 of Mignotte [4]; here, $h(\alpha)$ denotes the absolute logarithmic Weil height of α, defined, for an algebraic integer α, by

$$h(\alpha) = \frac{1}{[\mathbf{Q}(\alpha) : \mathbf{Q}]} \log \Pi_\sigma \max\{1, |\sigma(\alpha)|\}$$

where σ runs over the embedding of $\mathbf{Q}(\alpha)$ into $\mathbf{C}$.

Proposition 2 *Consider the linear form*

$$\Lambda = b_2 \log \alpha_2 - b_1 \log \alpha_1$$

where b_1 and b_2 are positive integers and α_1, α_2 are *nonzero, multiplicatively independent algebraic numbers. Set*

$$D = [\mathbf{Q}(\alpha_1,\ \alpha_2) : \mathbf{Q}]/[\mathbf{R}(\alpha_1,\ \alpha_2) : \mathbf{R}]$$

and let $\rho,\ \lambda,\ a_1$ *and* a_2 *be positive real numbers with* $\rho \geq 4,\ \lambda = \log \rho,$

$$a_i \geq \max\{1,\ \rho|\log \alpha_i| - \log|\alpha_i| + 2Dh(\alpha_i)\}\ (1 \leq i \leq 2)$$

and

$$a_1 a_2 \geq \max\{20,\ 4\lambda^2\}.$$

Further suppose h *is a real number with*

$$h \geq \max\left\{3.5,\ 1.5\lambda,\ D\left(\log\left(\frac{b_1}{a_2} + \frac{b_2}{a_1}\right) + \log \lambda + 1.377\right) + 0.023\right\},$$

$\chi = \dfrac{h}{\lambda},\ v = 4\chi + 4 + \dfrac{1}{\chi}$. *We may conclude, then, that*

$$\log|\Lambda| \geq -(C_0 + 0.06)(\lambda + h)^2\, a_1 a_2$$

where $C_0 = \dfrac{1}{\lambda^3}\left\{\left(2 + \dfrac{1}{2\chi(\chi+1)}\right)\left(\dfrac{1}{3} + \sqrt{\dfrac{1}{9} + \dfrac{4\lambda}{3v}\left(\dfrac{1}{a_1} + \dfrac{1}{a_2}\right) + \dfrac{32\sqrt{2}(1+\chi)^{\frac{3}{2}}}{3v^2\sqrt{a_1 a_2}}}\ \right)\right\}^2.$

3 Preparation toward the Proof of Theorem

M. A. Bennett proved in [1] that the equation (1) with $a,\, b,\, c$ positive integers and $a,\, b \geq 2$ has at most two solutions in positive integers $(x_i,\, y_i)\,(i = 1,\, 2)$, where $x_1 < x_2$ and $y_1 < y_2$.

Let us write

$$\Lambda_i = x_i \log a - y_i \log b\,(i = 1,\, 2).$$

According to [1](3.2),(3.3), we get

$$\log \Lambda_2 < \log \frac{c}{b^{y_2}} < \log \frac{ac}{(a-1)a^{x_2}}, \tag{2}$$

$$\frac{x_2}{\log b} > \frac{y_2}{\log a} > \frac{1}{\Lambda_1}. \tag{3}$$

Also we know([1] the bottom of p901),

$$\log \Lambda_i < \log \frac{c}{b^{y_i}}\ (i = 1,\, 2). \tag{4}$$

In the case of $\gcd(a,\, b) = 1$, we obtain $y_2 - y_1 > y_1$ through the second formula of (3.5) in [1]. That is $y_2 \geq 2y_1 + 1$. We use this inequality without previous notice.

4 The Case gcd(a, b)=1

Suppose first that $\gcd(a,\, b) = 1$ and that we have two positive solutions $(x_1,\, y_1)$ and $(x_2,\, y_2)$ to (1), with $x_1 < x_2$ and $y_1 < y_2$.

We apply Proposition 2 to

$$\Lambda_2 = x_2 \log a - y_2 \log b$$

where, in the notation of Proposition 2, we have

$$D = 1,\ \alpha_1 = b,\ \alpha_2 = a,\ b_1 = y_2,\ b_2 = x_2$$

and, since we assume $b \geq 2$ and $a \geq 6$, may take

$$a_1 = (\rho+1)\log b,\ a_2 = (\rho+1)\log a.$$

Choosing $\rho = 5.11$, it followes that $a_1 a_2 \geq 6.11^2 \log 2 \log 6 = 46.36471\ldots$ and $4\lambda^2 = 4(\log 5.11)^2 = 10.6432$ and $(\rho+1)\log 2 > 1$.
Therefore we can use Proposition 2 to these $a_1,\, a_2,\, b_1,\, b_2,\, \rho$.

Let us write

$$\log\left(\frac{y_2}{(\rho+1)\log a} + \frac{x_2}{(\rho+1)\log b}\right) + \log \lambda + 1.377 + 0.023 = \Omega.$$

As $a^{x_2} > b^{y_2}$ means $x_2 \log a > y_2 \log b$, we have $\dfrac{x_2}{\log b} > \dfrac{y_2}{\log a}$. From this inequality and $\log \lambda = \log(\log 5.11) = 0.489316\cdots$, we obtain

$$\Omega < \log \frac{x_2}{\log b} + \log 2 - \log(\rho+1) + 1.4 + \log(\log 5.11) = \log \frac{x_2}{\log b} + 0.7725\cdots < \log \frac{x_2}{\log b} + 0.773.$$

Let

$$h = \max\left\{8.56,\ \log\left(\frac{x_2}{\log b}\right) + 0.773\right\}.$$

That this is a valid choice for h follows from the above inequality.

5 $\mathbf{h = \log \frac{x_2}{\log b} + 0.773}$

In this case we have

$$\log \frac{x_2}{\log b} + 0.773 \geq 8.56.$$

Then

$$\frac{x_2}{\log b} \ge \exp 7.787 = 2409.07\ldots > 2409. \tag{5}$$

$\frac{1}{a_1} + \frac{1}{a_2}$ and $\frac{1}{a_1 a_2}$ are maximal for $(a,\, b)$
$= (7,\, 2)$.
Since $v = 4\chi + 4 + \frac{1}{\chi}$ is an increasing function for $\chi = \frac{h}{\lambda} \ge \frac{8.56}{\log 5.11} \ge 5.24767\cdots$, v is minimal for $\chi = 5.24767\cdots$ and $\frac{1}{v}$ is maximal for the same value of χ.
We have

$$\frac{1}{v^2}(1+\chi)^{\frac{3}{2}} = \frac{1}{v^{\frac{1}{2}}}\left(\frac{1+\chi}{4\chi+4+\frac{1}{\chi}}\right)^{\frac{3}{2}}.$$

As $\frac{1+\chi}{4\chi+4+\frac{1}{\chi}}$ is an increasing function for $\chi > 0$, its maximal value is $\frac{1}{4}$.
With these, from Proposition 2, we find that

$$C_0 < 0.556501\cdots < 0.5566$$

and we also have

$$\begin{aligned}&\log \Lambda_2 > -(0.5566 + 0.06)\\ &\times\left(\log 5.11 + \log\left(\frac{x_2}{\log b}\right) + 0.773\right)^2\\ &\times (5.11+1)^2 \log a \log b.\end{aligned}$$

We conclude

$$\begin{aligned}&\log \Lambda_2 > -23.01898\left(\log\left(\frac{x_2}{\log b}\right) + 2.405\right)^2\\ &\times \log a \log b.\end{aligned}$$

Combining this with (2), from $a \ge 6$, we obtain

$$\begin{aligned}&-x_2 \log a + \log\frac{6c}{5} \ge \log\frac{ac}{(a-1)a^{x_2}}\\ &> -23.01898\left(\log\left(\frac{x_2}{\log b}\right) + 2.405\right)^2 \log a \log b.\end{aligned}$$

Since $1 \le c \le 300$ and $\log a \log b \ge \log 7 \log 2$, we have

$$\begin{aligned}&\frac{x_2}{\log b} < \frac{\log\frac{6c}{5}}{\log a \log b}\\ &+ 23.01898\left(\log\left(\frac{x_2}{\log b}\right) + 2.405\right)^2\end{aligned}$$

contradicting (5).

6 The Case h = 8.56

In this case we have $\frac{x_2}{\log b} < 2410$.

Combining this with (3), (4), we find that

$$\frac{b^{y_1}}{c} < \frac{x_2}{\log b} < 2410. \tag{6}$$

For each value of $1 \le c \le 300$, this provides an upper bound upon b^{y_1} and , via $a^{x_1} = b^{y_1} + c$, upon a^{x_1},
i.e. $b^{y_1} < 2410 \times 300 = 723000,\ \ a^{x_1} < 723300$.

To complete the proof of Theorem for relatively prime a and b, we will argue as in Section 3 of [1].

Let us suppose that

$$\frac{b^{y_2}\log a}{c\, y_2} > 2, \tag{7}$$

so that $\frac{x_2}{y_2}$ is a convergent in simple continued fraction expansion to $\frac{\log b}{\log a}$, say $\frac{x_2}{y_2} = \frac{p_r}{q_r}$, where $\frac{p_r}{q_r}$ is the r-th such convergent.
In fact, we must have $x_2 = p_r$ and $y_2 = q_r$. If not, then $\gcd(x_2,\, y_2) = d > 1$ and so, writing $x_2 = dx$ and $y_2 = dy$,

$$a^{x_2} - b^{y_2} = (a^x - b^y)\sum_{i=0}^{d-1} a^{ix} b^{(d-i-1)y} = c.$$

It follows that

$$\sum_{i=0}^{d-1} a^{ix} b^{(d-i-1)y} \le c. \tag{8}$$

If $x_1 = 1$, this is a contradiction, since $a > a - b^{y_1} = c$.
If $x_1 = 2$, we have $x_2 \ge 3$.
If $d - 1 = 1$, we have $d = 2$, $x_2 = dx = 2x \ge 3$.
Therefore $x \ge 2$ and, so $(d-1)x \ge 2$. This is a contradiction, since $c \ge a^{(d-1)x} > a^2 - b^{y_1} = c$.
We may thus assume that $x_1 \ge 3$ (so that $x_2 \ge 4$).
If $d = 2$ and $x_2 = 4$, we have $y_2 \ge 6$, for $\gcd(x_2,\, y_2) = 2$ and $y_2 \ge 2y_1 + 1 \ge 3$.
Then (8) implies that $a^2 + b^3 \le c \le 300$.
Since we assume that a and b are not perfect powers, with $\gcd(a,\, b) = 1$, we find $6 \le a \le 17$. Now $a^4 - b^{y_2} = c \le 300$.
If $a = 6, a^4 = 6^4 = 1296$ and $b \ge 5$. This contradicts with $0 < 1296 - 5^{y_2} \le 300$, for $y_2 \ge 6$.
Similarly we have a contradiction in the cases of $7 \le a \le 17$.
Thus we conclude that we don't have the cases of $d = 2$, $x_2 = 4$, $x_1 = 3$, $y_2 \ge 6$.
If $d = 2$, $x_2 = 6$, $y_2 = 4$, we have $a^6 - b^4 = c$, i.e. $(a^3 - b^2)(a^3 + b^2) = c$.
If $a = 6$, then $9 \ge b \ge 5$, and so $a^3 + b^2 \ge 216 + 25 \ge 241$, $a^3 - b^2 \ge 216 - 81 = 135$. If $a \ge 7$, then $a^3 + b^2 > 343$.
These are a contradiction with $c \le 300$.
Similarly we find that the cases of $d = 2$, $x_2 \ge 6$, $y_2 \ge 4$ don't occur.
If $d = 3$, we obtain $x \ge 2$ from $x_2 \ge 4$. Then (8) means

$$b^{2y} + a^x b^y + a^{2x} \leq 300.$$

As $a \geq 6$, $a^{2x} \geq a^4 \geq 6^4 > 300$. This is a contradiction. Similarly if $d \geq 4$, we have a contradiction.
Eventually if $d > 1$, we find a contradiction.

If $\frac{p_r}{q_r}$ is the r-th convergent in the simple continued expansion to $\frac{\log b}{\log a}$, combining (2) with

$$\left|\frac{\log b}{\log a} - \frac{p_r}{q_r}\right| > \frac{1}{(a_{r+1}+2)q_r^2} \text{ (Khinchin [3])}$$

we have

$$\frac{c}{q_r b^{y_2} \log a} > \frac{\Lambda_2}{q_r \log a} = \left|\frac{\log b}{\log a} - \frac{p_r}{q_r}\right|$$

$$> \frac{1}{(a_{r+1}+2)q_r^2}$$

where a_{r+1} is the $(r+1)$-st partial quotient to $\frac{\log b}{\log a}$.
We thus have

$$a_{r+1} > \frac{b^{q_r} \log a}{c q_r} - 2. \tag{9}$$

For each pair (a, b) under consideration, we compute the initial terms in the simple continued expansion to $\frac{\log b}{\log a}$ via softwair ubasic and check to see if there exist a convergent $\frac{p_r}{q_r}$ with $p_r < 2410 \log b$, $p_r \geq 2$, $q_r \geq 3$ and related partial quotient a_{r+1} satisfiying (9) and $a^{p_r} - b^{q_r} = c$.
As $p_r = x_2$, $q_r = y_2$ and $a^{x_2} - b^{y_2} = a^{x_1} - b^{y_1} = c$, first we check the case that its smaller solution satisfies $a - b = c$.
We make b move from 2 to 723000, and c from 1 to 300.
We determine a from $a = b + c$.
If $\gcd(a, b) > 1$, we exclude such (a, b). Also if a or b are perfect powers we exclude them. If a is prime and the smallest $t \in \mathrm{N}$ such that $b^t \equiv 1(\mathrm{mod} a)$ is even, we also exclude them.
We obtain

a	b	c	p_r	q_r
13	3	10	3	7
91	2	89	2	13.

Under the condition $a = b + c$, these are only candidates that the corresponding equations have at most two solutions.
Next we investigate the case that its smaller solution satisfies $a - b^k = c(k \geq 2)$. Then we find that there is no (a, b, c) which has two solutions.
Similarly we check the case that its smaller solution satisfies $a^m - b = c(m \geq 2)$, and we know that there is no (a, b, c) which the corrsponding equation has two solutions.
At last we search the case that its smaller solution satisfies $a^m - b^k = c(m \geq 2,\ k \geq 2)$.
As candidates having two solutions , we have

a	b	c	p_r	q_r
56	5	11	2	5
130	7	93	2	5
57	5	124	2	5
58	5	239	2	5
47	3	22	2	7
23	2	17	2	9
421	3	94	2	11
47	2	161	2	11
91	2	89	2	13
13	3	10	3	7
6	23	95	7	4

It is easily cheked that among these, there exist positive integers $x_1 < p_r$, and $y_1 < q_r$ with

$$a^{p_r} - b^{q_r} = a^{x_1} - b^{y_1} > 0$$

only for $(a, b, c) = (91, 2, 89)$ and $= (13, 3, 10)$.
In fact we have $91^2 - 2^{13} = 91 - 2 = 89$, $13^3 - 3^7 = 13 - 3 = 10$.

Let us suppose that

$$\frac{b^{y_2} \log a}{c\, y_2} \leq 2. \tag{10}$$

Since $a \geq 6$, $y_2 \geq 3$ and $1 \leq c \leq 300$, we thus have $2 \leq b \leq 10$.
If $b = 10$, it follows that $a \geq 7$, $y_2 = 3$, $y_1 = 1$, whereby a^{x_1} divises 99. This implies that $c \leq 89$, contradicting (10).
If $b = 7$, it follows that $y_2 \geq 3$ from (10). If $y_2 \geq 4$, these cases contradict (10). Then $y_2 = 3$ and $y_1 = 1$. So we have $a^{x_2} - 7^3 = a^{x_1} - 7 = c$. Therefore $a^{x_1} | 7^2 - 1 = 48$. So $c \leq 41$, contradicting (10).
Similarly we can treat the cases of $b = 5, 6$.
If $b = 3$, $a \geq 10$ by Proposition 1. Since $c \leq 300$, we find $y_2 \leq 6$ by (10).
Then we have possibilities $y_1 = 1$ and $y_2 = 3, 4, 5, 6$; and $y_1 = 2$ and $y_2 = 5, 6$.
If $y_1 = 1$, $y_2 = 3$, $a^{x_1} | 3^2 - 1 = 8$ and $c \leq 5$ contradicting (10).
Similarly we find that the case $y_1 = 1$, $y_2 = 4$, contradicts (10).
If $y_1 = 1$, $y_2 = 5$, $a^{x_1} | 3^4 - 1 = 80$, so $c \leq 77$. The cases $a = 80, 40$ contradict (10). If $a = 20$, then $x_1 = 1$,

and from $20^{x_2} - 3^5 = 20 - 3 = 17$ we have $20^{x_2} = 260$. Contradiction. Similarly we find that the case $a = 10$ does not occur.
If $b = 2$, $a \geq 7$ from $\gcd(a, b) = 1$. Since $c \leq 300$, we have $y_2 \leq 11$ from (10). Then we have possibilities $y_1 = 1$ and $y_2 = 3, 4, 5, 6, 7, 8, 9, 10, 11$; and $y_1 = 2$ and $y_2 = 5, 6, 7, 8, 9, 10, 11$; and $y_1 = 3$ and $y_2 = 7, 8, 9, 10, 11$; and $y_1 = 4$ and $y_2 = 9, 10, 11$; and $y_1 = 5$ and $y_2 = 11$; We can treat these cases like the cases of $b = 3$, and we find that these cases do not occur.
Eventually we obtain that the case $\dfrac{b^{y_2} \log a}{c\, y_2} \leq 2\,(c \leq 300)$ does not occur.

Thus, with the result by Bennett(See 1 Introduction), we proved Theorem in the case of $\gcd(a, b) = 1$.

7 The case a,b are not relatively prime

If the equation at hand has two positive solutions, then from

$$a^{x_1}(a^{x_2-x_1} - 1) = b^{y_1}(b^{y_2-y_1} - 1),$$

if $\mathrm{ord}_p a = \alpha$ and $\mathrm{ord}_p b = \beta$, we find that

$$x_1\alpha = y_1\beta. \tag{11}$$

Also we have

$$\mathrm{ord}_p c = x_2\alpha. \tag{12}$$

To see this, we assume $x_2\alpha \geq y_2\beta$. From (11), we obtain $x_1y_2\alpha\beta \leq x_2y_1\alpha\beta$.
As $\alpha, \beta > 0$, this contradicts with $y_2x_1 > x_2y_1$([1](3.1)).
So $x_2\alpha < y_2\beta$. Combining this with $a^{x_2} - b^{y_2} = c$, we have (12).
By the just above inequality,

$$y_2\beta \geq x_2\alpha + 1. \tag{13}$$

Since $y_2x_1 > x_2y_1$, the equation

$$(b^{y_1} + c)^{\frac{x_2}{x_1}} - b^{y_2} = c \tag{14}$$

provides explicit bounds upon b and, via $a^{x_1} = b^{y_1} + c$, upon a.
We note that we need not consider squarefree values of c from (12) and $x_2 \geq 2$.

By way of example, we will give our arguments in detail for $c = 128 = 2^7$, $196 = 2^2 \times 7^2$, and $275 = 5^2 \times 11$.

If $c = 128 = 2^7$, from (12), $p = 2$, $x_2 = 7$, $\alpha = 1$. This implies $x_1 = 1, 2, 3, 4, 5, 6$.
If $x_1 = 1$, equation (11) means $y_1 = \beta = 1$, and so, from (13), (14)

$$(b + 128)^7 - b^8 \geq 128.$$

This implies $b \leq 126$. We make b move from 2 to 126 with step 2.
On each b, we check whether $(b+128)^7 - 128$ is a power of b. We find that in all the cases $(b+128)^7 - 128$ is not a power of b.
If $x_1 = 2$, equation (11) means $x_1\alpha = 2 = y_1\beta$, and so, $y_1 = 1$, $\beta = 2$ or $y_1 = 2$, $\beta = 1$.
In the first case, from (13), (14) we are led to

$$(b + 128)^{\frac{7}{2}} - b^4 \geq 128$$

whereby $b \leq 126$. We make b move from 2 to 126 with step 2.
First we check whether $(b+128)^{\frac{7}{2}} - 128$ is an integer or not. If this is an integer we check whether it is a power of b. We find that in all the cases $(b+128)^{\frac{7}{2}} - 128$ is not a power of b.
Similarly we can treat the second case, and also we conclude that this case does not occur.
From similar deduction, we find that the equation $a^x - b^y = 128$ has at most one positive solution (x, y).

If $c = 196 = 2^2 \times 7^2$, first we check the case $p = 7$. We have $x_2\alpha = 2$, so $x_2 = 2, \alpha = 1$ and $x_1 = 1$ and from (11), $y_1 = \beta = 1$. So we are led to

$$(b + 196)^2 - b^3 \geq 196$$

whereby $b \leq 35$.
We make b move from 7 to 35 with step 7. On each b, we check whether $(b + 196)^2 - 196$ is a power of b. We find that in all the cases $(b+196)^2 - 196$ is not a power of b.
Second $p = 2$. Then we have $(b + 196)^2 - b^3 \geq 196$, whereby $b \leq 34$. From similar argument we have that $(b+196)^2 - 196$ is not a power of b where $b = 2, 4, 6, \cdots, 34$.
So we find that the equation $a^x - b^y = 196$ has at most one positive solution (x, y).

If $c = 275 = 5^2 \times 11$, we have $p = 5$, $x_2\alpha = 2$, so, $x_2 = 2$, $\alpha = 1$, and from (11), $y_1 = \beta = 1$.
So we are led to

$$(b + 275)^2 - b^3 \geq 275$$

whereby $b \leq 45$.
We make b move from 5 to 45 with step 5.
We only have $280^2 - 5^7 = 275$. It is easy to see that the smaller solution of $280^x - 5^y = 275$ is $(1, 1)$.

Arguing similarly for the remaining non-squarefree values of $c \leq 300$, together with the result by Bennett(See 1

Introduction), we find that there are no other additional triples (a, b, c) with $\gcd(a, b) > 1$ for which (1) has two positive solutions.

This completes the proof of Therorem.

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