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PROPAGATION FORMULA FOR PRINCIPAL SERIES WHITTAKER FUNCTIONS ON $GL(3, \mathbf{C})$

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ABSTRACT: In this paper, we give an expression of principal series Whittaker functions on $GL(3, \mathbf{C})$ in terms of those on $GL(2, \mathbf{C})$. This is an analogous formula to the one in the real class one case disscussed by Ishii and Stade [5].

Keywords: Automorphic forms, Automorphic representations, Whittaker functions.

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1. Introduction

Automorphic forms and representations are important objects in number theory and their investigations are mainly based on the properties of associated special functions. In particular, the precise analytic properties of Whittaker functions (see §2 for the definition) are required in the study of the Fourier expansions of automorphic forms and their related topics such as L-functions (*cf.* [2] and its references).

A propagation formula for class one Whittaker functions on $GL(n, \mathbf{R})$ was given in the recent paper of Ishii-Stade [5]. This is a formula expressing Whittaker functions on $GL(n, \mathbf{R})$ in terms of those on $GL(n-1, \mathbf{R})$ and the proof is based on their explicit formulas. In the class one case, there is an elementary relation between real and complex Whittaker functions on $GL(n)$ [9], and thus we have a propagation formula for those on $GL(n, \mathbf{C})$. We expect that there is a similar propagation formula in the non-class one cases. However in such cases, the relation between real and complex Whittaker functions is not known and also their explicit formulas are not obtained yet except some cases of low degrees (*cf.* [6], [7], [4]).

In this paper, we give a propagation formula for principal series Whittaker functions on $GL(3, \mathbf{C})$ based on their explicit formulas obtained in our recent paper [4]. Our result seems not only to show the similarity between the real and the complex cases but also to give a hint on a basis of $\mathfrak{gl}_n$-module which is suitable for an explicit description of the Whittaker functions on $GL(n, \mathbf{C})$ (*cf.* [1], [3]).

2. Definition of Whittaker functions

In this section, we recall the definition of Whittaker function which is our main object in this paper.

Let $G = NAK$ be an Iwasawa decomposition of a real reductive group G. For an (irreducible) admissible representation (π, H_π) of G, we choose a K-type (τ^*, V_{τ^*}) in π which occurs with multiplicity one and fix an injective K-homomorphism $i \in \mathrm{Hom}_K(\tau^*, \pi|_K)$. Here (τ^*, V_{τ^*}) means the contragradient representation of (τ, V_τ). Moreover, take a non-degenerate character η of N. Let us consider the intertwining space

$$\mathcal{I}_{\eta,\pi} = \mathrm{Hom}_{(\mathfrak{g}_\mathbf{C}, K)}(\pi, C^\infty \mathrm{Ind}_N^G(\eta))$$

between $(\mathfrak{g}_\mathbf{C}, K)$-modules π and $C^\infty\mathrm{Ind}_N^G(\eta)$ consisting of all K-finite vectors, where $C^\infty\mathrm{Ind}_N^G(\eta)$ is the induced representation of G from η as C^∞-induction. For each $T \in \mathcal{I}_{\eta,\pi}$, we define a V_τ-valued function T_i on G by

$$T(i(v^*))(g) = \langle v^*, T_i(g)\rangle, \ \ v^* \in V_{\tau^*}, \ \ g \in G.$$

Here $\langle\cdot,\cdot\rangle$ is the canonical bilinear form on $V_{\tau^*} \times V_\tau$. The function T_i means a restriction of $T \in \mathcal{I}_{\eta,\pi}$ to K and satisfies

$$T_i(ngk) = \eta(n)\tau(k)^{-1}T_i(g), \quad (n, g, k) \in N \times G \times K.$$

Then we put

$$\begin{aligned} &\mathrm{Wh}(\pi, \eta, \tau)^{\mathrm{mod}} \\ = \ &\bigcup_i \{T_i \mid T \in \mathcal{I}_{\eta,\pi},\, T_i \text{ is moderate growth}\}. \end{aligned}$$

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Here, the union runs through all embeddings $i \in \mathrm{Hom}_K(\tau^*, \pi|_K)$ and the term "moderate growth" is by means of [10]. According to the multiplicity one theorem of Shalika [8], the dimension of the space $\mathrm{Wh}(\pi,\eta,\tau)^{\mathrm{mod}}$ is at most one. A unique (up to constant) element in $\mathrm{Wh}(\pi,\eta,\tau)^{\mathrm{mod}}$ is called a *(primary) Whittaker function*.

3. Whittaker functions on $GL(3,\mathbf{C})$

In this section, we recall an explicit formula of principal series Whittaker functions on $GL(3,\mathbf{C})$ obtained in our previous paper [4].

3.1 Groups and representations

Let $G = GL(3,\mathbf{C})$ be the complex general linear group of degree 3, which is viewed as a real reductive group, with the center

$$Z_G = \{ru1_3 \mid r \in \mathbf{R}_{>0}, u \in U(1)\} \simeq \mathbf{C}^\times.$$

Here 1_n is the unit matrix of degree n. Let $K = U(3)$ be a maximal compact subgroup of G, and define subgroups A and N of G by

$$A = \{\mathrm{diag}\,(a_1,a_2,a_3) \in G \mid a_i \in \mathbf{R}_{>0}, i = 1,2,3\},$$

$$N = \left\{ n(\boldsymbol{x}) = \begin{pmatrix} 1 & x_1 & x_2 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{pmatrix} \in G \;\middle|\; \boldsymbol{x} = (x_i) \in \mathbf{C}^3 \right\}.$$

Then we have an Iwasawa decomposition $G = NAK$. The centralizer M of A in K is given by

$$\begin{aligned} M &= \{\mathrm{diag}\,(u_1,u_2,u_3) \mid u_i \in U(1),\, i = 1,2,3\} \\ &\simeq U(1)^3. \end{aligned}$$

Then $P = NAM$ is the upper triangular subgroup of G, which is a minimal parabolic subgroup of G.

The equivalence classes of irreducible continuous representations of K are parameterized by the set of highest weights

$$\Lambda = \{\mu = (\mu_1,\mu_2,\mu_3) \mid \mu \in \mathbf{Z}^3, \mu_1 \geq \mu_2 \geq \mu_3\}.$$

We denote by (τ_μ, V_μ) the representation of K associated with $\mu \in \Lambda$ and take the (normalized) GZ-basis $\{f(M)\}_{M \in G(\mu)}$ of the representation space V_μ which is parameterized by the set $G(\mu)$ of G-patterns M belonging to μ. Here a G-pattern $M \in G(\mu)$ is a triangle

$$M = \begin{pmatrix} \mu_1 \;\mu_2\; \mu_3 \\ \alpha_1 \;\alpha_2 \\ \beta \end{pmatrix}$$

consisting of 6 integers satisfying the inequalities

$$\mu_1 \geq \alpha_1 \geq \mu_2 \geq \alpha_2 \geq \mu_3, \quad \alpha_1 \geq \beta \geq \alpha_2.$$

Let us take a character $\sigma_{\mathbf{n}}$ of M defined by

$$\sigma_{\mathbf{n}}(\mathrm{diag}(u_1,u_2,u_3)) = u_1^{n_1}u_2^{n_2}u_3^{n_3},$$

with the parameter $\mathbf{n} = (n_1,n_2,n_3) \in \mathbf{Z}^3$. Moreover, if we denote the complexification of the Lie algebra of A by $\mathfrak{a}_{\mathbb{C}}$ and the diagonal matrix unit with (i,i)-entry 1 and the remaining entries 0 by $E_{ii} \in \mathfrak{a}_{\mathbb{C}}$, let us take an element ν in the dual $\mathfrak{a}_{\mathbb{C}}^*$ of $\mathfrak{a}_{\mathbb{C}}$ identified with $(\nu_1,\nu_2,\nu_3) \in \mathbf{C}^3$ via $\nu_i = \nu(E_{ii})$ for $1 \leq i \leq 3$. Then the induced representation

$$\pi = \pi(\nu,\sigma_{\mathbf{n}}) = \mathrm{Ind}_P^G(1_N \otimes e^{\nu+\rho} \otimes \sigma_{\mathbf{n}})$$

of G from the parabolic subgroup $P = NAM$ is called the *principal series representation* of G. Here ρ is the half-sum of the positive restricted roots, i.e.,

$$e^\rho(\mathrm{diag}\,(a_1,a_2,a_3)) = \left(\frac{a_1}{a_3}\right)^2, \ \mathrm{diag}\,(a_1,a_2,a_3) \in A.$$

The central character of π is given by

$$Z_G \ni ru1_3 \mapsto r^{\tilde{\nu}}u^{\tilde{n}}, \quad r \in \mathbf{R}_{>0},\ u \in U(1),$$

with $\tilde{\nu} = \nu_1+\nu_2+\nu_3$ and $\tilde{n} = n_1+n_2+n_3$, and the minimal K-type of π is the representation $(\tau_{\mathbf{m}}, V_{\mathbf{m}})$ of K associated with the dominant permutation $\mathbf{m} \in \Lambda$ of $\mathbf{n}$.

We take a non-degenerate character η of N defined by

$$\eta(n(\boldsymbol{x})) = \exp\left(2\pi\sqrt{-1}\mathrm{Im}\,(x_1+x_3)\right).$$

3.2 Explicit formula

Let $\pi = \pi(\nu,\sigma_{\mathbf{n}})$ be an *irreducible* principal series representation with the minimal K-type $(\tau^*, V_{\tau^*}) = (\tau_{\mathbf{m}}, V_{\mathbf{m}})$ associated with the dominant permutation $\mathbf{m} = (m_1,m_2,m_3) \in \Lambda$ of $\mathbf{n}$, and let η be the non-degenerate unitary character of N defined in the previous subsection.

For an element $f(M)$ in the GZ-basis $\{f(M)\}$ of $V_{\mathbf{m}}$ and a Whittaker function $\phi \in \mathrm{Wh}(\pi,\eta,\tau)^{\mathrm{mod}}$ which is an V_τ-valued function on G, we define the M-component $\phi(M)$ of ϕ by

$$\phi(M;g) = \langle \phi(g), f(M) \rangle, \quad g \in G.$$

Whittaker functions are determined by its A-radial parts (i.e. its restriction to A) because of the Iwasawa decomposition of G. Moreover, the values of Whittaker functions on the center Z_G of G are given by the central character of π, i.e.,

$$\phi(rug) = r^{\tilde{\nu}}u^{\tilde{n}}\phi(g),$$

where $r \in \mathbf{R}_{>0}$, $u \in U(1)$, and $g \in G$. Therefore, we can describe Whittaker functions as functions of two variables with the coordinates

$$y_1 = \frac{a_1}{a_2}, \quad y_2 = \frac{a_2}{a_3}$$

for $\mathrm{diag}\,(a_1,a_2,a_3) = a_3 \cdot \mathrm{diag}\,(y_1y_2, y_2, 1) \in A$.

To state an explicit formula for the primary Whittaker function on G, we introduce some notations. If we write $\mathbf{m} = (n_a, n_b, n_c)$, then we put

$$(\lambda_1, \lambda_2, \lambda_3) = \left(\nu_c - \frac{\tilde{\nu}}{3}, \nu_a - \frac{\tilde{\nu}}{3}, \nu_b - \frac{\tilde{\nu}}{3}\right).$$

For each G-pattern $M = \left(\begin{smallmatrix} m_1 & m_2 & m_3 \\ \alpha_1 & \alpha_2 & \\ & \beta & \end{smallmatrix} \right) \in G(\mathbf{m})$, we put $\delta(M) = \alpha_1 + \alpha_2 - m_2 - \beta$ and

$$\begin{aligned}
\zeta_1^{(1)}(M) &= \lambda_1 - m_3 + \beta, \\
\zeta_2^{(1)}(M) &= \lambda_2 + m_1 - \beta, \\
\zeta_3^{(1)}(M) &= \lambda_3 + \alpha_1 - \alpha_2 - |\delta(M)|, \\
\zeta_1^{(2)}(M) &= -\lambda_1 + m_1 - \beta - \delta(M), \\
\zeta_2^{(2)}(M) &= -\lambda_2 - m_3 + \beta + \delta(M), \\
\zeta_3^{(2)}(M) &= -\lambda_3 + m_1 - m_3 - \alpha_1 + \alpha_2.
\end{aligned}$$

Theorem 1. *Let $W_3(y) \in \mathrm{Wh}(\pi, \eta, \tau)^{\mathrm{mod}}$ be the (A-radial part of) primary Whittaker function with the M-components $W_3(M; y) = y_1^2 y_2^2 \tilde{W}_3(M; y)$ for each G-pattern $M = \left(\begin{smallmatrix} m_1 & m_2 & m_3 \\ \alpha_1 & \alpha_2 & \\ & \beta & \end{smallmatrix} \right) \in G(\mathbf{m})$. Then the function $\tilde{W}_3(M; y)$ has the following integral expressions:*

$$\begin{aligned}
& \tilde{W}_3(M; y) \\
= \; & \frac{1}{(2\pi\sqrt{-1})^2} \\
& \times \int_{s_1} \int_{s_2} V_3(M; s_1, s_2) (\pi y_1)^{-s_1} (\pi y_2)^{-s_2} ds_1 ds_2 \\
= \; & 2^4 (\pi y_1)^{\frac{-\lambda_3 + m_1 - m_3}{2}} (\pi y_2)^{\frac{\lambda_3 + m_1 - m_3}{2}} \\
& \times \int_0^\infty K_A\left(2\pi y_1 \sqrt{1 + \frac{1}{v}}\right) K_B\left(2\pi y_2 \sqrt{1+v}\right) \\
& \times v^C (1+v)^D \frac{dv}{v}.
\end{aligned}$$

Here, in the first integral expression of Mellin-Barnes type, the paths s_i of integrations are the vertical lines from $\mathrm{Re}\, s_i - \sqrt{-1}\infty$ to $\mathrm{Re}\, s_i + \sqrt{-1}\infty$ with enough large real part and the integrand $V_3(M; s_1, s_2)$ is defined by

$$\begin{aligned}
& V_3(M; s_1, s_2) \\
= \; & \prod_{i=1}^{2} \prod_{j=1}^{3} \Gamma\left(\frac{s_i + \zeta_j^{(i)}(M)}{2}\right) \\
& \times \Gamma\left(\frac{s_1 + s_2 + \zeta_3^{(1)}(M) + \zeta_3^{(2)}(M)}{2}\right)^{-1}.
\end{aligned}$$

Also, in the second integral expression of Euler type, K_ν is the K-Bessel function and the parameters A, B, C, and D are given by

$$\begin{aligned}
A &= \frac{\zeta_1^{(1)}(M) - \zeta_2^{(1)}(M)}{2}, \\
B &= A + \delta(M), \\
C &= \frac{2\zeta_3^{(1)}(M) - \zeta_1^{(1)}(M) - \zeta_2^{(1)}(M)}{4}, \\
D &= \frac{|\delta(M)|}{2}.
\end{aligned}$$

4. Whittaker functions on $GL(2, \mathbf{C})$

In this section, we derive an explicit formula of principal series Whittaker functions on $GL(2, \mathbf{C})$ by similar computation to the case of $GL(3, \mathbf{C})$.

4.1 Groups and representations

Let $G' = GL(2, \mathbf{C})$ be the complex general linear group of degree 2 and $G' = N'A'K'$ be its Iwasawa decomposition, where $K' = U(2)$ is a maximal compact subgroup of G' and

$$A' = \left\{ \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \,\middle|\, a_i \in \mathbf{R}_{>0}, i = 1, 2 \right\},$$

$$N' = \left\{ n(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \,\middle|\, x \in \mathbf{C} \right\}.$$

The center $Z_{G'}$ of G' is $\{ru1_2 \mid r \in \mathbf{R}_{>0}, u \in U(1)\} \simeq \mathbf{C}^\times$. The upper triangular subgroup of G' is $P' = N'A'M'$, where M' is the centralizer of A' in K' given by

$$M' = \left\{ \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} \,\middle|\, u_i \in U(1),\, i = 1, 2 \right\} \simeq U(1)^2.$$

Let $\mathfrak{g}' = \mathfrak{gl}(2, \mathbf{C})$ be the Lie algebra of G'. If we put a Cartan involution $\theta(X) = -{}^t\bar{X}$ for $X \in \mathfrak{g}'$ and denote the $+1$ and the -1 eigenspaces of θ in $\mathfrak{g}'$ by $\mathfrak{k}'$ and $\mathfrak{p}'$, respectively. Then $\mathfrak{k}'$ is the Lie algebra of K' and $\mathfrak{g}'$ has a Cartan decomposition $\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{p}'$. For $1 \le i, j \le 2$, let E_{ij} (resp. E'_{ij}) in $\mathfrak{g}'$ be the matrix unit with its (i,j)-entry 1 (resp. J) and the remaining entries 0. Here J is the imaginary unit; $J^2 = -1$. Moreover put $H_{12} = E_{11} - E_{22}$, $H'_{12} = E'_{11} - E'_{22}$, $I_2 = E_{11} + E_{22}$, and $I'_2 = E'_{11} + E'_{22}$. Then we have $\mathfrak{k}' = Z_{\mathfrak{k}'} \oplus \mathfrak{k}'_0$ and $\mathfrak{p}' = Z_{\mathfrak{p}'} \oplus \mathfrak{p}'_0$ with

$$\begin{aligned}
Z_{\mathfrak{k}'} &= \mathbf{R}I'_2, \\
\mathfrak{k}'_0 &= \mathbf{R}H'_{12} \oplus \mathbf{R}(E_{12} - E_{21}) \oplus \mathbf{R}(E'_{12} + E'_{21}), \\
Z_{\mathfrak{p}'} &= \mathbf{R}I_2, \\
\mathfrak{p}'_0 &= \mathbf{R}H_{12} \oplus \mathbf{R}(E_{12} + E_{21}) \oplus \mathbf{R}(E'_{12} - E'_{21}).
\end{aligned}$$

In the complexifications $\mathfrak{k}'_{\mathbb{C}}$ and $\mathfrak{p}'_{\mathbb{C}}$, we use the symbols $I_2^{\mathfrak{k}'} = -\sqrt{-1}I'_2$, $H_{12}^{\mathfrak{k}'} = \sqrt{-1}H'_{12}$, and

$$E_{ij}^{\mathfrak{k}'} = \frac{1}{2}\left\{(E_{ij} - E_{ji}) - \sqrt{-1}\left(E'_{ij} + E'_{ji}\right)\right\},$$

in $\mathfrak{k}'_{\mathbb{C}}$ and $I_2^{\mathfrak{p}'} = I_2$, $H_{12}^{\mathfrak{p}'} = H_{12}$, and

$$E_{ij}^{\mathfrak{p}'} = \frac{1}{2}\left\{(E_{ij}+E_{ji}) - \sqrt{-1}\left(E'_{ij}-E'_{ji}\right)\right\},$$

in $\mathfrak{p}'_{\mathbb{C}}$.

We can parameterize the equivalence classes of irreducible continuous representations of $K' = U(2)$ by the set

$$\Lambda' = \{\mu' = (\mu'_1, \mu'_2) \,|\, \mu' \in \mathbf{Z}^2, \mu'_1 \geq \mu'_2\},$$

from the highest weight theory. The representation space $V_{\mu'}$ of the representation $\tau_{\mu'}$ associated with $\mu' = (\mu'_1, \mu'_2) \in \Lambda'$ has the (normalized) GZ-basis $\{f'(M')\}_{M' \in G(\mu')}$ as in the case of $U(3)$. Here

$$G(\mu') = \left\{ M' = \left(\begin{smallmatrix} \mu'_1 & \mu'_2 \\ \multicolumn{2}{c}{\alpha'} \end{smallmatrix} \right) \middle| \, \alpha' \in \mathbf{Z},\ \mu'_1 \geq \alpha' \geq \mu'_2 \right\}.$$

The explicit action of $\mathfrak{k}'_{\mathbb{C}}$ on the GZ-basis is given as follows.

$$\begin{aligned} E_{ii}^{\mathfrak{k}'} f'(M') &= w'_i f'(M), \qquad i = 1, 2, \\ E_{12}^{\mathfrak{k}'} f'(M') &= (\mu'_1 - \alpha') f'\left(M'(1)\right), \\ E_{21}^{\mathfrak{k}'} f'(M') &= (\alpha' - \mu'_2) f'\left(M'(-1)\right). \end{aligned}$$

Here $(w'_1, w'_2) = (\alpha', \mu'_1 + \mu'_2 - \alpha')$ is the weight of the vector $f'(M')$ associated with a G-pattern $M' = \left(\begin{smallmatrix} \mu'_1 & \mu'_2 \\ \multicolumn{2}{c}{\alpha'} \end{smallmatrix} \right)$ and $M'(i) = \left(\begin{smallmatrix} \mu'_1 & \mu'_2 \\ \multicolumn{2}{c}{\alpha'+i} \end{smallmatrix} \right)$. Moreover, we promise the corresponding vector $f'(M')$ is zero if $M'(i)$ appearing in the above formulas violates the conditions of G-patterns.

A principal series representation

$$\pi' = \pi'(\nu', \sigma_{\mathbf{n}'}) = \mathrm{Ind}_{P'}^{G'}(1_{N'} \otimes e^{\nu'+\rho'} \otimes \sigma_{\mathbf{n}'}),$$

of G' with data $\nu' = (\nu'_1, \nu'_2) \in \mathbf{C}^2$ and $\mathbf{n}' = (n'_1, n'_2) \in \mathbf{Z}^2$ induced from the minimal parabolic subgroup $P' = N'A'M'$ is defined similarly to that of $GL(3, \mathbf{C})$. In this case, the half-sum ρ' of the positive restricted roots is given by

$$e^{\rho'}(\mathrm{diag}\,(a_1, a_2)) = \frac{a_1}{a_2}, \quad \mathrm{diag}\,(a_1, a_2) \in A'.$$

As in the case of $GL(3, \mathbf{C})$, the central character of π' is

$$Z_{G'} \ni ru1_2 \mapsto r^{\tilde{\nu}'} u^{\tilde{n}'}, \quad r \in \mathbf{R}_{>0},\ u \in U(1),$$

with $\tilde{\nu}' = \nu'_1 + \nu'_2$ and $\tilde{n}' = n'_1 + n'_2$, and the minimal K'-type of π' is the representation $(\tau_{\mathbf{m}'}, V_{\mathbf{m}'})$ associated with the dominant permutation $\mathbf{m}' \in \Lambda'$ of $\mathbf{n}'$.

Finally, we take a non-degenerate character η' of N' defined by

$$\eta'(n(x)) = \exp\left(2\pi\sqrt{-1}\mathrm{Im}\,(x)\right).$$

4.2 Differential equations

Let $\pi' = \pi'(\nu', \sigma_{\mathbf{n}'})$ be an *irreducible* principal series representation of G' with the minimal K'-type $(\tau_{\mathbf{m}'}, V_{\mathbf{m}'})$ associated with the dominant permutation $\mathbf{m}' = (m'_1, m'_2) \in \Lambda'$ of $\mathbf{n}'$, and let η' be a non-degenerate unitary character of N' defined in the previous subsection.

It is well known that an element $\mathcal{C}$ in the center $Z(\mathfrak{g}'_{\mathbb{C}})$ of the universal enveloping algebra $U(\mathfrak{g}'_{\mathbb{C}})$ of $\mathfrak{g}'_{\mathbb{C}}$ acts as a scalar on the K'-finite vectors in π'. Thus, each M'-component $\phi(M')$ of a Whittaker function $\phi \in \mathrm{Wh}(\pi', \eta', \tau')^{\mathrm{mod}}$ which is defined similarly to the case of $GL(3, \mathbf{C})$ satisfies a differential equation

$$\mathcal{C}\phi(M') = \chi_{\mathcal{C}}\phi(M') \tag{1}$$

with an eigenvalue $\chi_{\mathcal{C}}$. The next lemma gives generators of $Z(\mathfrak{g}'_{\mathbb{C}})$ constructed from the Capelli elements in $U(\mathfrak{g}')$ via the identification of $U(\mathfrak{g}'_{\mathbb{C}})$ and $U(\mathfrak{g}') \otimes_{\mathbb{C}} U(\mathfrak{g}')$, and the eigenvalue $\chi_{\mathcal{C}}$ of $\mathcal{C}$. These are obtained by the same way as in [4].

Lemma 2. 1. *The following four elements $Cp_k^{(i)}$ in $U(\mathfrak{g}'_{\mathbb{C}})$ give an independent generators of $Z(\mathfrak{g}'_{\mathbb{C}})$.*

$$\begin{aligned} Cp_1^{(1)} &= \frac{1}{2}\left(I_2^{\mathfrak{p}'} + I_2^{\mathfrak{k}'}\right), \\ Cp_1^{(2)} &= \frac{1}{2}\left(I_2^{\mathfrak{p}'} - I_2^{\mathfrak{k}'}\right), \\ Cp_2^{(1)} &= \frac{1}{4}\Big\{\left(E_{11}^{\mathfrak{p}'} + E_{11}^{\mathfrak{k}'} - 1\right)\left(E_{22}^{\mathfrak{p}'} + E_{22}^{\mathfrak{k}'} + 1\right) \\ &\qquad - \left(E_{12}^{\mathfrak{p}'} + E_{12}^{\mathfrak{k}'}\right)\left(E_{21}^{\mathfrak{p}'} + E_{21}^{\mathfrak{k}'}\right)\Big\}, \\ Cp_2^{(2)} &= \frac{1}{4}\Big\{\left(E_{11}^{\mathfrak{p}'} - E_{11}^{\mathfrak{k}'} - 1\right)\left(E_{22}^{\mathfrak{p}'} - E_{22}^{\mathfrak{k}'} + 1\right) \\ &\qquad - \left(E_{12}^{\mathfrak{p}'} - E_{12}^{\mathfrak{k}'}\right)\left(E_{21}^{\mathfrak{p}'} - E_{21}^{\mathfrak{k}'}\right)\Big\}. \end{aligned}$$

2. *The eigenvalues $\chi_{Cp_k^{(i)}}$ of the generators $Cp_k^{(i)}$ are given as follows.*

$$\begin{aligned} \chi_{Cp_1^{(1)}} &= \frac{1}{2}\{(\nu'_1 + n'_1) + (\nu'_2 + n'_2)\}, \\ \chi_{Cp_1^{(2)}} &= \frac{1}{2}\{(\nu'_1 - n'_1) + (\nu'_2 + n'_2)\}, \\ \chi_{Cp_2^{(1)}} &= \frac{1}{4}(\nu'_1 + n'_1)(\nu'_2 + n'_2), \\ \chi_{Cp_2^{(2)}} &= \frac{1}{4}(\nu'_1 - n'_1)(\nu'_2 - n'_2). \end{aligned}$$

By virtue of the Iwasawa decomposition $G' = N'A'K'$ of G' and the central character of π', we can describe Whittaker functions as functions of a variable

$$y = \frac{a_1}{a_2}, \quad \text{for } \mathrm{diag}\,(a_1, a_2) = a_2 \cdot \mathrm{diag}\,(y, 1) \in A'.$$

We denote the Euler operator with respect to y by $\partial = y\dfrac{\partial}{\partial y}$.

To obtain the explicit description of the differential equation (1), we need the following fundamental lemma.

Lemma 3. *Let $\phi = \phi(x) \in \mathrm{Wh}(\pi', \eta', \tau')^{\mathrm{mod}}$.*

1. *The actions of elements $H_{12}^{\mathfrak{p}'}$, and $I_2^{\mathfrak{p}'}$ in $\mathfrak{a}'_{\mathbb{C}}$ on ϕ are*
$$H_{12}^{\mathfrak{p}'}\phi = 2\partial\phi,\ \ I_2^{\mathfrak{p}'}\phi = \tilde{\nu}'\phi.$$
Thus, for $E_{ii}^{\mathfrak{p}'}$ we have
$$E_{11}^{\mathfrak{p}'}\phi = \left(\partial + \frac{\tilde{\nu}'}{2}\right)\phi,\ \ E_{22}^{\mathfrak{p}'}\phi\left(-\partial + \frac{\tilde{\nu}'}{2}\right)\phi.$$
2. *The actions of elements $E_{12}^{\mathfrak{p}'} + E_{12}^{\mathfrak{k}'}$ and $E_{21}^{\mathfrak{p}'} - E_{21}^{\mathfrak{k}'}$ in $\mathfrak{n}'_{\mathbb{C}}$ on ϕ are the following multiplications.*

$$\left(E_{21}^{\mathfrak{p}'} - E_{21}^{\mathfrak{k}'}\right)\phi = -2\pi y\phi,\ \ \left(E_{12}^{\mathfrak{p}'} + E_{12}^{\mathfrak{k}'}\right)\phi = 2\pi y\phi.$$

Computing the actions of the generators $Cp_k^{(i)}$ by Lemma 3, we can write the differential equations (1) explicitly as in the next proposition.

Proposition 4. *Let $\phi(M')$ be the M'-component of a Whittaker function $\phi \in \mathrm{Wh}(\pi', \eta', \tau')^{\mathrm{mod}}$ and put $\phi(M'; y) = y\tilde{\phi}(M'; y)$. Then the differential equations (1) for the Capelli elements $\mathcal{C} = Cp_2^{(i)}$ with $i = 1, 2$ are given as follows: Let $(w'_1, w'_2) = (\alpha', m'_1 + m'_2 - \alpha')$ be the weight of a G-pattern $M' = \left(\begin{smallmatrix} m'_1 & & m'_2 \\ & \alpha' & \end{smallmatrix}\right)$.*

1. *For $\mathcal{C} = Cp_2^{(1)}$, we have*
$$\left[\left(\partial + \frac{\tilde{\nu}'}{2} + w'_1\right)\left(-\partial + \frac{\tilde{\nu}'}{2} + w'_2\right)\right.$$
$$\left. - \left(2\pi\sqrt{-1}\right)^2 y^2 - (\nu'_1 + n'_1)(\nu'_2 + n'_2)\right]\tilde{\phi}(M'; y)$$
$$-4\pi y\,(\alpha' - m'_2)\,\tilde{\phi}\,(M'(-1); y) = 0.$$
2. *For $\mathcal{C} = Cp_2^{(2)}$, we have*
$$\left[\left(\partial + \frac{\tilde{\nu}'}{2} - w'_1\right)\left(-\partial + \frac{\tilde{\nu}'}{2} - w'_2\right)\right.$$
$$\left. - \left(2\pi\sqrt{-1}\right)^2 y^2 - (\nu'_1 - n'_1)(\nu'_2 - n'_2)\right]\tilde{\phi}(M'; y)$$
$$-4\pi y\,(m'_1 - \alpha')\,\tilde{\phi}\,(M'(1); y) = 0.$$

In particular, the equation for $\mathcal{C} = Cp_2^{(2)}$ at the G-pattern $L' = \left(\begin{smallmatrix} m'_1 & & m'_2 \\ & m'_1 & \end{smallmatrix}\right)$ associated with the highest weight vector $f'(L')$ in $V_{\mathbf{m}'}$ gives the following differential equation for $\tilde{\phi}(L')$.

$$\left[\left(\partial + \frac{\tilde{\nu}'}{2} - m'_1\right)\left(-\partial + \frac{\tilde{\nu}'}{2} - m'_2\right)\right.$$
$$\left. - \left(2\pi\sqrt{-1}\right)^2 y^2 - (\nu'_1 - n'_1)(\nu'_2 - n'_2)\right]\tilde{\phi}(L'; y)$$
$$= 0.$$

If we put
$$\lambda'_1 = \nu'_b - \frac{\tilde{\nu}'}{2},\ \ \lambda'_2 = \nu'_a - \frac{\tilde{\nu}'}{2}$$
for $\mathbf{m}' = (m'_1, m'_2) = (n'_a, n'_b)$, then we have the relations $\lambda'_1 + \lambda'_2 = 0$ and
$$(\nu'_1 \pm n'_1)\,(\nu'_2 \pm n'_2)$$
$$= \left(\lambda'_2 + \frac{\tilde{\nu}'}{2} \pm m'_1\right)\left(\lambda'_1 + \frac{\tilde{\nu}'}{2} \pm m'_2\right),$$
and thus we can write the above equation for $\tilde{\phi}(L')$ as
$$\left[\partial^2 - (m'_1 - m'_2)(\partial + \lambda'_1)\right.$$
$$\left. + \lambda'_1\lambda'_2 + \left(2\pi\sqrt{-1}\right)^2 y^2\right]\tilde{\phi}(L'; y) = 0.$$

4.3 Explicit formula

The M'-components of the primary Whittaker function are given as a moderate growth solution of the differential equations in Proposition 4. The explicit formula for them is given in the next theorem.

Theorem 5. *Let $W_2(x) \in \mathrm{Wh}(\pi', \eta', \tau')^{\mathrm{mod}}$ be the (A'-radial part of) primary Whittaker function with the M'-components $W_2(M'; y) = y\tilde{W}_2(M'; y)$ for each G-pattern $M' = \left(\begin{smallmatrix} m'_1 & & m'_2 \\ & \alpha' & \end{smallmatrix}\right) \in G(\mathbf{m}')$. Then the function $\tilde{W}_2(M'; y)$ has the following integral expression:*
$$\begin{aligned}\tilde{W}_2(M'; y) &= \frac{1}{2\pi\sqrt{-1}}\int_s V_2(M'; s)(\pi y)^{-s} ds \\ &= 4(\pi y)^A K_B(2\pi y).\end{aligned}$$
Here, the path of integration is the vertical line from $\mathrm{Re}\, s - \sqrt{-1}\infty$ to $\mathrm{Re}\, s + \sqrt{-1}\infty$ with enough large real part and the integrand $V_2(M'; s)$ is defined by
$$\begin{aligned}V_2(M'; s) &= \Gamma\left(\frac{s + \lambda'_2 + m'_1 - \alpha'}{2}\right) \\ &\quad \times\Gamma\left(\frac{s + \lambda'_1 + \alpha' - m'_2}{2}\right),\end{aligned}$$
and the parameters A and B are given by
$$A = \frac{m'_1 - m'_2}{2},\ \ B = \frac{\lambda'_1 - \lambda'_2 + w'_1 - w'_2}{2}.$$

5. Propagation formula

In this section, we give an expression of Whittaker functions on $GL(3, \mathbf{C})$ in terms of those on $GL(2, \mathbf{C})$, which is an analogue of the formula obtained by Ishii-Stade [5].

5.1 Preliminaries

Here we recall some formulas which are fundamental in this section.

The modified Bessel function $K_\nu(z)$ of the second kind has several integral expressions. Among them, we need two expressions: One is the integral expression of Mellin-Barnes type

$$\begin{aligned} K_\nu(z) &= \frac{1}{4}\cdot\frac{1}{2\pi\sqrt{-1}} \\ &\quad \times\int_s \Gamma\left(\frac{s+\nu}{2}\right)\Gamma\left(\frac{s-\nu}{2}\right)\left(\frac{z}{2}\right)^{-s} ds. \end{aligned}$$

Here, the path of integration is the vertical line from $\mathrm{Re}\, s-\sqrt{-1}\infty$ to $\mathrm{Re}\, s+\sqrt{-1}\infty$ with enough large real part. Another is that of Euler type

$$K_\nu(z) = \frac{1}{2}\times\int_0^\infty \exp\left(\frac{-z(t+t^{-1})}{2}\right) t^\nu \frac{dt}{t},$$

which is valid only for $\mathrm{Re}\, z>0$.

Also we need the following integral formula so-called Barnes' lemma

$$\begin{aligned} &\frac{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+b+c+d)} \\ =\ &\frac{1}{2\pi\sqrt{-1}} \\ &\times\int_z \Gamma(z+a)\Gamma(z+b)\Gamma(-z+c)\Gamma(-z+d)dz. \end{aligned}$$

Here the path of integration is the vertical line from $\mathrm{Re}\, z-\sqrt{-1}\infty$ to $\mathrm{Re}\, z+\sqrt{-1}\infty$ with enough large real part.

5.2 Main theorem

Let $\pi=\pi(\nu,\sigma_{\mathbf{n}})$ be an irreducible principal series representation of $G=GL(3,\mathbf{C})$ with data $\nu=(\nu_1,\nu_2,\nu_3)\in\mathbf{C}^3$ and $\mathbf{n}=(n_1,n_2,n_3)\in\mathbf{Z}^3$ and let η be a non-degenerate unitary character of N defined in §2. For simplicity, we assume that the parameter $\mathbf{n}$ satisfies the regularity condition

$$n_1\geq n_2\geq n_3.$$

Then π has the minimal K-type $(\tau_{\mathbf{m}},V_{\mathbf{m}})=(\tau_{\mathbf{n}},V_{\mathbf{n}})$.

Let $W_3(y)\in \mathrm{Wh}(\pi,\eta,\tau)^{\mathrm{mod}}$ be the (primary) Whittaker function, and for each G-pattern $M=\begin{pmatrix} m_1\, m_2\, m_3 \\ \alpha_1\, \alpha_2 \\ \beta \end{pmatrix}\in G(\mathbf{m})$ denote its M-component by $W_3(M;y)=y_1^2y_2^2\tilde{W}_3(M;y)$. Under the regularity condition on $\mathbf{n}$, we have the parameters

$$(\lambda_1,\lambda_2,\lambda_3)=\left(\nu_3-\frac{\tilde{\nu}}{3},\nu_1-\frac{\tilde{\nu}}{3},\nu_2-\frac{\tilde{\nu}}{3}\right),$$

which appear in the integrand $V_3(M;s_1,s_2)$ of the integral expression for $\tilde{W}_3(M;y)$ of Mellin-Barnes type in Theorem 1.

Theorem 6. *The integrand $V_3(M;s_1,s_2)$ has the following expression.*

$$\begin{aligned} &V_3(M;s_1,s_2) \\ =\ &\Gamma\left(\frac{s_1+\zeta_j^{(1)}(M)}{2}\right)\Gamma\left(\frac{s_2+\zeta_j^{(2)}(M)}{2}\right) \\ &\times\frac{1}{2\pi\sqrt{-1}}\int_z V_2(M';-z) \\ &\times\Gamma\left(\frac{z+s_1+\mu_1}{2}\right)\Gamma\left(\frac{z+s_2+\mu_2}{2}\right)dz, \end{aligned}$$

where $V_2(M';s)$ is the integrand of the integral expression of $\tilde{W}_2(M';y)$ in Theorem 5 for a triple $(\pi'(\nu',\sigma_{\mathbf{n}'}),\eta',\tau_{\mathbf{m}'})$ and a G-pattern $M'\in G(\mathbf{m}')$. The parameters and the representations are given as follows.

1. *If $\delta(M)\geq 0$, we have $j=2$ and*
$$\mu_1=-\frac{\lambda_2}{2}+\beta-\alpha_2,\ \ \mu_2=\frac{\lambda_2}{2}+m_1-\alpha_1,$$
$$\nu'=(\nu_2,\nu_3),\ \ \mathbf{n}'=\mathbf{m}'=(m_2,m_3),$$
$$M'=\begin{pmatrix} m_2\, m_3 \\ \alpha_2 \end{pmatrix}.$$
2. *If $\delta(M)\leq 0$, we have $j=1$ and*
$$\mu_1=-\frac{\lambda_1}{2}+\alpha_1-\beta,\ \ \mu_2=\frac{\lambda_1}{2}+\alpha_2-m_3,$$
$$\nu'=(\nu_1,\nu_2),\ \ \mathbf{n}'=\mathbf{m}'=(m_1,m_2),$$
$$M'=\begin{pmatrix} m_1\, m_2 \\ \alpha_1 \end{pmatrix}.$$

Proof. Assume $\delta(M)\geq 0$. Then, since $\zeta_1^{(1)}(M)+\zeta_1^{(2)}(M)=\zeta_3^{(1)}(M)+\zeta_3^{(2)}(M)$, Barnes' lemma leads the equation

$$\begin{aligned} &V_3(M;s_1,s_2) \\ =\ &\Gamma\left(\frac{s_1+\zeta_2^{(1)}(M)}{2}\right)\Gamma\left(\frac{s_2+\zeta_2^{(2)}(M)}{2}\right) \\ &\times\frac{1}{2\pi\sqrt{-1}}\int_z \Gamma\left(\frac{-z+\mu_3}{2}\right)\Gamma\left(\frac{-z+\mu_4}{2}\right) \\ &\times\Gamma\left(\frac{z+s_1+\mu_1}{2}\right)\Gamma\left(\frac{z+s_2+\mu_2}{2}\right)dz, \end{aligned}$$

where the parameters μ_1 and μ_2 are given in the assertion of theorem and μ_3 and μ_4 are

$$\mu_3=\frac{-\nu_2+\nu_3}{2}+\alpha_2-m_3,\quad \mu_4=\frac{\nu_2-\nu_3}{2}-\alpha_2+m_2.$$

Here we use the relations $\lambda_1+\dfrac{\lambda_2}{2}=\dfrac{-\nu_2+\nu_3}{2}$ and $\lambda_3+\dfrac{\lambda_2}{2}=\dfrac{\nu_2-\nu_3}{2}$.

In the case of $\delta(M)\leq 0$, the relation $\zeta_2^{(1)}(M)+\zeta_2^{(2)}(M)=\zeta_3^{(1)}(M)+\zeta_3^{(2)}(M)$ brings the assertion by similar computation. □

Corollary 7. *We have the following expression of* $\tilde{W}_3(M;y)$.

$$\tilde{W}_3(M;y) = \frac{2^4}{2\pi\sqrt{-1}}\int_z (\pi y_1)^{a_1+\frac{z}{2}} K_{A_1-\frac{z}{2}}(2\pi y_1) \times(\pi y_2)^{a_2+\frac{z}{2}} K_{-A_2+\frac{z}{2}}(2\pi y_2) \times V_2(M';-z)dz.$$

Here

$$a_k = \frac{1}{2}\left\{\zeta_j^{(k)}(M)+\mu_k\right\}, \quad A_k = \zeta_j^{(k)}(M) - a_k,$$

for $k = 1,2$, *and the parameters and the representations are given in Theorem 6.*

Proof. Using the first integral expression of $K_\nu(z)$ of Mellin-Barnes type in §4.1, we can get the corollary from Theorem 6 together with the integral expression of Mellin-Barnes type for $\tilde{W}_3(M;y)$ in Theorem 1. □

Corollary 8. *We have the following expression of* $\tilde{W}_3(M;y)$.

$$\begin{aligned}&\tilde{W}_3(M;y)\\ =\ & 4\pi^{a_1+a_2} y_1^{a_1+A_1} y_2^{a_2-A_2}\\ &\times\int_0^\infty\int_0^\infty u_1^{A_1}u_2^{-A_2}\\ &\times\exp\left(-\pi\left(y_1^2u_1+\frac{1}{u_1}+y_2^2u_2+\frac{1}{u_2}\right)\right)\\ &\times\tilde{W}_2\left(M';\pi y_2\sqrt{\frac{u_2}{u_1}}\right)\frac{du_1}{u_1}\frac{du_2}{u_2}.\end{aligned}$$

Here the parameters and the representations are given in Theorem 6.

Proof. By applying the second integral expression of $K_\nu(z)$ of Euler type in §4.1 to the expression of $\tilde{W}_3(M;y)$ in Corollary 7, we have

$$\begin{aligned}&\tilde{W}_3(M;y)\\ =\ & \frac{4}{2\pi\sqrt{-1}}\int_0^\infty\int_0^\infty\int_z\\ &\times\exp\left(-\pi y_1\left(u_1+\frac{1}{u_1}\right)\right)u_1^{A_1}\\ &\times\exp\left(-\pi y_2\left(u_2+\frac{1}{u_2}\right)\right)u_2^{-A_2}\\ &\times(\pi y_1)^{a_1}(\pi y_2)^{a_2}\left(\pi^2 y_1y_2\frac{u_2}{u_1}\right)^{\frac{z}{2}} V_2(M';-z)\\ &\times\frac{du_1}{u_1}\frac{du_2}{u_2}dz.\end{aligned}$$

Then we can get the assertion by the substitutions $u_1 \to u_1y_1$, $u_2 \to u_2y_2$, and $z \to -z$ in the above integrals. □

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