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A REMARK ON ARCHIMEDEAN ZETA INTEGRALS ON $GL(n) \times GL(m)$

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Abstract: We explicitly compute the archimedean zeta integrals for $GL(n) \times GL(m)$ with $m = n, n-1, n-2$ via explicit formula of local Whittaker functions on $GL(n)$.

Keywords: automorphic L-functions, zeta integrals, Whittaker functions

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Introduction

In his paper in 1937, Hecke proved fundamental facts (analytic continuation, functional equation and converse theorem) on a Dirichlet series attached to an elliptic cusp form by using integral representation. He also expressed the Dirichlet series as Euler product (=standard L-functions on $GL(2)$).

Soon after Rankin and Selberg independently obtained an integral representation for the standard L-functions on $GL(2) \times GL(2)$. After that various kinds of integral representations for automorphic L-functions have been found by many people up to now, and the way of studying automorphic L-functions via integral representations are called Rankin-Selberg method (cf. [3]).

The first step of Rankin-Selberg method is to find an appropriate global zeta integral and show a 'basic identity,' that is, to express the (global) zeta integral in terms of certain integral transformation of global spherical functions, such as Whittaker functions.

When the global spherical function is decomposed into the product of local spherical functions, the global zeta integral also can be written as a product of local zeta integrals. Then our investigation is reduced to the study of the local zeta integrals.

The next step is the local analysis at the unramified places. In many cases, the local spherical functions at the unramified places are explicitly given (e.g. Kato, Casselman-Shalika formulas for Whittaker functions). By using them, the coincidence of the local zeta integrals and the local L-factors can be shown.

To obtain global results, we need to study the local integrals at bad places. However, to control the local zeta integrals at the ramified places and the archimedean places are very difficult problem. Then even the functional equations of automorphic L-functions have established in very few cases.

Our aim here is to study archimedean zeta integrals by means of the explicit formulas of spherical functions. From that motivation many spherical functions on the real symplectic group $Sp(2, \mathbb{R})$ have been studied these 15 years around T. Oda.

On the other hand, study of the class one Whittaker functions on the group $GL(n, \mathbb{R})$ have been proceeded by Bump [2], Stade (e.g. [7], [8]) and the author [5] and so on. Stade and the author [4] reached a recursive formula with respect to n and it seems to be a 'natural' formula.

In this paper we study archimedean zeta integrals for the standard L-function on $GL(n) \times GL(m)$ by using the explicit formula in [4]. We will compute explicitly the archimedean zeta integrals when the archimedean components are isomorphic to the class one principal series representations. Stade [7], [8] proved that the archimedean zeta integrals coincide with the archimedean L-factors determined by Langlands parameter in the case of $n = m, m-1$. On the other hand, for $n > m-1$, these coincidences are not expected. We calculate the local zeta integral in the case of $n = m-2$ and determine the ratio of zeta integrals and L-factors.

1. Integral representations of L-functions on $GL(n) \times GL(m)$

We recall general theory of integral representations of standard L-functions on $GL(n) \times GL(m)$.

1.1. **Whittaker functions.** Let k be a number field and $\mathbb{A}$ its adele ring. Fix a nontrivial additive character ψ of $k\backslash\mathbb{A}$. Let $N = N_n$ be the standard maximal unipotent subgroup of GL_n consisting of upper triangular unipotent matrices. We denote by ψ_N the character of $N_n(\mathbb{A})$ defined by

$$\psi_N(x) = \psi(x_{12} + \cdots + x_{n-1,n})$$

for $x = (x_{ij}) \in N_n(\mathbb{A})$.

Let π be a cuspidal automorphic representation of $GL_n(\mathbb{A})$. A cusp form $\varphi \in \pi$ has a Fourier

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expansion

$$\varphi(g) = \sum_{\gamma \in N_{n-1}(k)\backslash GL_{n-1}(k)} W_\varphi\left(\begin{pmatrix}\gamma & \\ & 1\end{pmatrix} g\right)$$

with the global Whittaker function

$$W_\varphi(g) = \int_{N_n(k)\backslash N_n(\mathbb{A})} \varphi(xg)\psi_N(x)^{-1}\, dx.$$

We call the space $\mathcal{W}(\pi, \psi) = \{W_\varphi \mid \varphi \in \pi\}$ the Whittaker model of π.

Let π_v be an irreducible admissible representation of $GL_n(k_v)$. We extend a nontrivial character ψ_v of k_v to the character $\psi_{N,v}$ of $N_n(k_v)$ and set

$$\begin{aligned}\mathcal{W}(\psi_v) = \{W : GL_n(k_v) \to \mathbb{C} \text{ smooth} \\ \mid W(ng) = \psi_{N,v}(n)W(g)\},\end{aligned}$$

on which the group $GL_n(k_v)$ acts on via right translations. We denote by $\mathcal{W}(\pi_v, \psi_v)$ the space of the image of nontrivial intertwiner $\pi_v \hookrightarrow \mathcal{W}(\psi_v)$, and call it the Whittaker model of π_v.

For a vector ξ in π_v, we call its image $W_\xi \in \mathcal{W}(\pi_v, \psi_v)$ the Whittaker function attached to ξ.

1.2. **Zeta integrals.** We recall the integral representations of (degree nm) standard L-functions on $GL(n) \times GL(m)$ $(n \geq m)$. Let $\pi = \otimes'_v \pi_v$ and $\pi' = \otimes'_v \pi'_v$ be cuspidal automorphic representations of $GL_n(\mathbb{A})$ and $GL_m(\mathbb{A})$, respectively. We explain the case of $n > m$. For cusp forms $\varphi \in \pi$, $\varphi' \in \pi'$ and $s \in \mathbb{C}$, we consider the zeta integral

$$\begin{aligned}I(s, \varphi, \varphi') = \int_{GL_m(k)\backslash GL_m(\mathbb{A})} (P\varphi)\begin{pmatrix} g & 0 \\ 0 & 1_{n-m}\end{pmatrix} \\ \times \varphi'(g)|\det g|^{s-1/2}\, dg,\end{aligned}$$

where P means a projection to GL_{m+1}:

$$\begin{aligned}(P\varphi)(h) = |\det h|^{(m-n+1)/2} \int_{X_{n,m}(k)\backslash X_{n,m}(\mathbb{A})} \\ \times \varphi\left(x\begin{pmatrix} h & 0 \\ 0 & 1_{n-m-1}\end{pmatrix}\right)\psi^{-1}(x)\, dx,\end{aligned}$$

with $X_{n,m}$ is the unipotent radical of the parabolic subgroup of GL_n corresponding to the partition $(m+1, 1, \ldots, 1)$ of n. We also define the zeta integral $\widetilde{I}$ by

$$\begin{aligned}\widetilde{I}(s, \varphi, \varphi') = \int_{GL_m(k)\backslash GL_m(\mathbb{A})} (\widetilde{P}\varphi)\begin{pmatrix} g & 0 \\ 0 & 1_{n-m}\end{pmatrix} \\ \times \varphi'(g)|\det g|^{s-1/2}\, dg,\end{aligned}$$

with $\widetilde{P} = \iota \circ P \circ \iota$. Here $\iota(g) = {}^t g^{-1}$. Then these integrals I and $\widetilde{I}$ are absolutely convergent for all s and satisfy the functional equation

$$I(s, \varphi, \varphi') = \widetilde{I}(1 - s, \widetilde{\varphi}, \widetilde{\varphi}'),$$

where $\tilde{\varphi}(g) = \varphi({}^t g^{-1})$ and φ belongs to the contragradient $\widetilde{\pi}$ of π.

By using the Fourier expansion, these integrals are unfolded to reach the basic identity

$$\begin{aligned}I(s, \varphi, \varphi') = \int_{N_m(\mathbb{A})\backslash GL_m(\mathbb{A})} W_\varphi\begin{pmatrix} g & 0 \\ 0 & 1_{n-m}\end{pmatrix} \\ \times W_{\varphi'}(g)\,|\det g|^{s-(n-m)/2}\, dg.\end{aligned}$$

If the cusp forms φ and φ' are decomposable in π and π', respectively, that is, $\varphi = \otimes_v \xi_v$ and $\varphi' = \otimes_v \xi'_v$, then the local uniqueness of Whittaker model implies that the Euler product expansions the global Whittaker functions and thus we get the Euler product expansions of zeta integrals:

$$\begin{aligned}I(s, \varphi, \varphi') &= \prod_v \Psi(s, W_{\xi_v}, W'_{\xi'_v}), \\ \widetilde{I}(s, \widetilde{\varphi}, \widetilde{\varphi}) &= \prod_v \widetilde{\Psi}(s, \rho(w_{n,m})\widetilde{W}_{\xi_v}, \widetilde{W}'_{\xi'_v}),\end{aligned}$$

where $\widetilde{W}_v(g) = W(w_n {}^t g^{-1}) \in \mathcal{W}(\widetilde{\pi}_v, \psi_v^{-1})$ and $\widetilde{W}'_v(g) = W(w_m {}^t g^{-1}) \in \mathcal{W}(\widetilde{\pi}'_v, \psi_v)$ with $w_n = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}$. Here $\rho(w_{n,m})$ is the right translation by $w_{n,m} = \begin{pmatrix} 1_n & \\ & w_{n-m}\end{pmatrix}$. Here the local zeta integrals are

$$\begin{aligned}\Psi(s, W, W') = \int_{N_n(k_v)\backslash GL_n(k_v)} W\begin{pmatrix} g & \\ & 1\end{pmatrix} \\ \times W'(g)\,|\det g\,|^{s-\frac{n-m}{2}} dg\end{aligned}$$

and

$$\begin{aligned}\widetilde{\Psi}(s, W, W') = \int_{N_n(k_v)\backslash GL_n(k_v)} \int_{M_{n-m-1,m}(k_v)} \\ \times W\begin{pmatrix} g & & \\ x & 1_{n-m-1} & \\ & & 1\end{pmatrix} W'(g) \\ \times\ |\det g\,|^{s-\frac{n-m}{2}} dx\, dg.\end{aligned}$$

We want to show the local functional equation

$$\begin{aligned}\frac{\widetilde{\Psi}(1-s, \rho(w_{n,m})\widetilde{W}_v, \widetilde{W}'_v)}{L(1-s, \widetilde{\pi}_v, \widetilde{\pi}'_v)} \\ = \varepsilon_v(s, \psi_v, \pi_v, \pi'_v) \cdot \frac{\Psi(s, W_v, W'_v)}{L(s, \pi_v, \pi'_v)}\end{aligned} \tag{1.1}$$

for appropriate L and ε factors. Then we arrive at the global functional equation

$$L(s, \pi, \pi') = \varepsilon(s, \pi, \pi')L(1 - s, \tilde{\pi}, \tilde{\pi}'),$$

where

$$\begin{aligned}L(s, \pi, \pi') &= \prod_v L(s, \pi_v, \pi'_v), \\ \varepsilon(s, \pi, \pi') &= \prod_v \varepsilon(s, \pi_v, \pi'_v).\end{aligned}$$

We note that the case of $n = m$, the global zeta integral contains an Eisenstein series on GL_n.

Including the case of $n=m$, Jacquet and Shalika proved the local functional equation (1.1) by induction arguments and non-vanishing of (1.1). Further they showed that $L(s,\pi_v,\pi'_v)$ can be written as a finite sum of $\Psi(s,W_v,W'_v)$ in case of $m=n,n-1$. It has not been expected such phenomenon for $m\le n-2$. Then to understand the ratio $\Psi(s,W_v,W'_v)/L(s,\pi_v,\pi'_v)$ more precisely, we compute it in the case of $m=n-2$.

2. Whittaker functions on $GL_n(\mathbb{R})$

In this section, we discuss explicit formulas of Whittaker functions belonging to the class one principal series of $GL_n(\mathbb{R})$.

2.1. **Class one principal series representations of $GL_n(\mathbb{R})$.** For $\nu\in\mathbb{C}$, let χ_ν be the character of $GL_1(\mathbb{R})$ such that $\chi_\nu(x)=|x|^\nu$. For $a=(a_1,\dots,a_n)\in\mathbb{C}^n$, the class one principal series π_a is the induced representation $\mathrm{Ind}_{B_n(\mathbb{R})}^{GL_n(\mathbb{R})}(\chi_{a_1}\boxtimes\cdots\boxtimes\chi_{a_n})$, where B_n is the standard Borel subgroup of GL_n.

Hereafter we consider the case of $v\cong\mathbb{R}$ and assume that $\pi_v\cong\pi_a$ and $\pi'_v\cong\pi_{a'}$, with $a=(a_1,\dots,a_n)\in\mathbb{C}^n$ and $a'=(a'_1,\dots,a'_m)\in\mathbb{C}^m$. We note that $\widetilde{\pi}_v\cong\pi_{-a}$ and $\widetilde{\pi}'_v\cong\pi_{-a'}$. The local L and ε factors determined by Langlands classification are

$$L(s,\pi_v,\pi'_v)=\prod_{1\le i\le n,1\le j\le m}\Gamma_{\mathbb{R}}(s+a_i+a'_j)$$

and

$$\varepsilon_v(s,\psi_v,\pi_v,\pi'_v)=1.$$

Here $\Gamma_{\mathbb{R}}(s)=\pi^{-s/2}\Gamma(s/2)$.

When $\pi_v\cong\pi_a$, there exists a (unique up to constant) spherical vector ξ_0 in π_v, and we call the Whittaker function W_{ξ_0} attached to ξ_0 the *class one Whittaker function*. Since the class one Whittaker function $W=W_{\xi_0}$ has the property $W(ngk)=\psi_{N,v}(n)W(g)$, for $(n,g,k)\in N_n(\mathbb{R})\times GL_n(\mathbb{R})\times O(n)$, the Iwasawa decomposition implies that the function W is determined by its restriction to

$$A_n(\mathbb{R})=\{\mathrm{diag}(a_1,\dots,a_n)\mid a_i>0,a_n=1\}.$$

We call the restriction $W|_{A_n(\mathbb{R})}$ the *radial part* of W. For simplicity, we may assume that the central characters of π and π' are trivial and this implies that

$$\sum_{i=1}^n a_i=0,\quad \sum_{i=1}^m a'_i=0,$$

for $\pi_v\cong\pi_a$ and $\pi'_v\cong\pi_{a'}$.

We introduce a coordinate $y=(y_1,\dots,y_{n-1})\in\mathbb{R}_+^{n-1}$ on A by $y_i=a_i/a_{i+1}$, and denote by

$$y^\rho\equiv y^{\rho_n}=\prod_{i=1}^{n-1}y_i^{n(n-i)/2}.$$

2.2. **Jacquet integral.** We recall Jacquet's integral representation of the class one Whittaker functions ([6]). The radial part $J_{n,a}(y)=y^\rho J^0_{n,a}(y)$ of Jacquet integral can be written as

(2.1)
$$\begin{aligned}J^0_{n,a}(y)&=\prod_{k=1}^{n-1}y_k^{-(a_1+\cdots+a_{n-k})}\int_{\mathbb{R}^{n(n-1)/2}}\\&\times\prod_{k=1}^{n-1}\left(\Delta_{n,k}((x_{ij}))\right)^{(a_{n-k+1}-a_{n-k}-1)/2}\\&\times\exp\Big(-2\pi\sqrt{-1}\sum_{k=1}^{n-1}y_kx_{k,k+1}\Big)\prod_{1\le i<j\le n}dx_{ij},\end{aligned}$$

where $\Delta_{n,k}\equiv\Delta_{n,k}(x)\equiv\Delta_{n,k}((x_{ij}))$ is the sum of squares of $k\times k$ subdeterminants of the $k\times n$ matrix formed by the top k rows of (x_{ij}). To relate $J_{n,a}(y)$ and $J_{n-1,a}(y)$, we start with a lemma which is easily shown.

Lemma 1. *For $1\le k\le n$, set*

$$q_k=\Delta_{k,1}=1+x_{12}^2+\cdots+x_{1k}^2$$

and

$$x'_{ij}=\sqrt{\frac{q_iq_{j-1}}{q_{i-1}q_j}}\left(x_{ij}-\sum_{a=i}^{j-1}\frac{x_{1a}x_{1j}}{q_{j-1}}x_{ia}\right).$$

Then we have

$$\begin{aligned}\Delta_{n,k}&\equiv\Delta_{n,k}((x_{ij}))\\&=\frac{q_n}{q_k}\cdot\Delta_{n-1,k-1}((x'_{i+1,j+1})_{1\le i<j\le n-1}).\end{aligned}$$

for $2\le k\le n-1$.

By means of this lemma, we substitute $x_{ij}\to x'_{ij}$ in (2.1) and denote x'_{ij} by x_{ij} again to arrive at

$$\begin{aligned}J^0_{n,a}(y)&=\prod_{k=1}^{n-1}y_k^{-(a_1+\cdots+a_{n-k})}\int_{\mathbb{R}^{n-1}}\prod_{j=2}^{n}dx_{1j}\\&\times\prod_{k=2}^{n-1}q_k^{\frac{a_{n-k}-a_{n-k+1}-1}{2}}\cdot q_n^{\frac{a_n-a_1-1}{2}}\\&\times\exp\left(-2\pi\sqrt{-1}\sum_{k=1}^{n-1}y_k\frac{x_{1k}x_{1,k+1}}{q_k}\right)\\&\times\int_{\mathbb{R}^{(n-1)(n-2)/2}}\prod_{2\le i<j\le n}dx_{ij}\\&\times\prod_{k=1}^{n-2}\left(\Delta_{n,k-1}((x_{i+1,j+1}))\right)^{\frac{a_{n-k}-a_{n-k-1}-1}{2}}\end{aligned}$$

$$\times \exp\Big(-2\pi\sqrt{-1}\sum_{k=2}^{n-1} y_k \frac{\sqrt{q_{k-1}q_{k+1}}}{q_k} x_{k,k+1}\Big).$$

If we set

$$\tilde{a} = (\tilde{a}_k)_{1\le k\le n-1}; \quad \tilde{a}_k = a_k + \frac{a_n}{n-1},$$

then we have $\sum_{k=1}^{n-1}\tilde{a}_k = 0$, $a_{n-k} - a_{n-k-1} = \tilde{a}_{n-k} - \tilde{a}_{n-k-1}$ and the inner integral $\int_{\mathbb{R}^{(n-1)(n-2)/2}}$ becomes

$$\prod_{k=2}^{n-1}\Big(y_k \frac{\sqrt{q_{k-1}q_{k+1}}}{q_k}\Big)^{\tilde{a}_1+\cdots+\tilde{a}_{n-k}}$$
$$\times J^0_{n-1,\tilde{a}}\Big(y_2\frac{\sqrt{q_3}}{q_2}, y_3\frac{\sqrt{q_2q_4}}{q_3}, \dots, y_{n-1}\frac{\sqrt{q_{n-2}q_n}}{q_{n-1}}\Big).$$

By collecting powers of y_k and q_k we can reach the following recursive relation for Jacquet integrals.

Lemma 2. *We have*

$$J^0_{n,a}(y) = \prod_{k=1}^{n-1} y_k^{\frac{n-k}{n-1}a_n} \int_{\mathbb{R}^{n-1}} \prod_{j=2}^{n} dx_{1j}$$
$$\times (q_2\cdots q_{n-1})^{-\frac12} q_n^{\frac{na_n}{2(n-1)}-\frac12}$$
$$\times \exp\Big(-2\pi\sqrt{-1}\sum_{k=1}^{n-1} y_k \frac{x_{1k}x_{1,k+1}}{q_k}\Big)$$
$$\times J^0_{n-1,\tilde{a}}\Big(y_2\frac{\sqrt{q_3}}{q_2}, y_3\frac{\sqrt{q_2q_4}}{q_3}, \dots, y_{n-1}\frac{\sqrt{q_{n-2}q_n}}{q_{n-1}}\Big).$$

2.3. Mellin-Barnes integral representations.

We recall the Mellin-Barnes integral representations of Whittaker functions obtained by Stade and the author ([4]). Set

$$W_{n,a}(y) = y^\rho W^0_{n,a}(y) := \Big(\prod_{1\le i<j\le n}\Gamma_{\mathbb{R}}(a_i - a_j + 1)\Big) J_{n,a}(y).$$

For $s = (s_1, \dots, s_{n-1}) \in \mathbb{C}^{n-1}$, let

$$T_{n,a}(s) \equiv T_{n,a}(s_1,\dots,s_{n-1}) = 2^{n-1}\int_{\mathbb{R}_+^{n-1}} W^0_{n,a}(y) \prod_{1\le i\le n-1}(\pi y_i)^{-s_i}\frac{dy_i}{y_i}$$

be the multiple Mellin transform of $W^0_{n,a}$. Then the Mellin inversion implies that

$$W_{n,a}(y) = \frac{y^\rho}{(4\pi\sqrt{-1})^{n-1}}\int_{s_1,\dots,s_{n-1}} \times T_{n,a}(s)\prod_{j=1}^{n-1}(\pi y_j)^{-s_j} ds_j.$$

with the path of integration in each s_j being a vertical line in the complex plane, of sufficiently large real part to keep the poles of $T_{n,a}(s)$ on its left. Note that

$$T_{2,a}(s) = \Gamma\Big(\frac{s+a_1}{2}\Big)\Gamma\Big(\frac{s+a_2}{2}\Big).$$

Stade and the author [4] obtained the following recursive relation between $T_{n,a}(s)$ and $T_{n-1,a}(s)$.

Proposition 3. [4]

$$\begin{aligned} T_{n,a}(s) &= \frac{1}{(4\pi\sqrt{-1})^{n-2}}\int_{z_1,\dots,z_{n-2}} \\ &\times \prod_{j=1}^{n-1}\Gamma\Big(\frac{s_j - z_{j-1}}{2} + \frac{(n-j)a_1}{2(n-1)}\Big) \\ &\times \prod_{j=1}^{n-1}\Gamma\Big(\frac{s_j - z_j}{2} - \frac{ja_1}{2(n-1)}\Big) \\ &\times T_{n-1,b}(z_1,\dots,z_{n-2})\prod_{j=1}^{n-2}dz_j, \end{aligned} \tag{2.2}$$

with

$$b = (b_i)_{1\le i\le n-1}; \quad b_i = a_{i+1} + \frac{a_1}{n-1}.$$

Here we understand $z_0 = z_{n-1} = 0$.

The next formulas are important in our evaluation of archimedean zeta integrals.

Proposition 4. *(a) For a complex number ν, we have*

$$\begin{aligned} T_{n,a}(s) &= \frac{\Gamma(\frac{s_{n-1}+\nu}{2})}{\prod_{j=1}^n\Gamma(\frac{a_j+\nu}{2})}\cdot\frac{1}{(4\pi\sqrt{-1})^{n-1}} \\ &\times \int_{z_1,\dots,z_{n-1}}\prod_{j=1}^{n-1}\Gamma\Big(\frac{s_j - z_j}{2}\Big) \\ &\times \prod_{j=1}^{n-1}\Gamma\Big(\frac{s_{j-1} - z_j + \nu}{2}\Big) \\ &\times T_{n,a}(z_1,\dots,z_{n-1})\prod_{j=1}^{n-1}dz_j. \end{aligned} \tag{2.3}$$

Here we understand $s_0 = 0$.
(b) For a complex number ν, we have

$$\begin{aligned} T_{n,a}(s) &= \frac{\Gamma(\frac{s_1+\nu}{2})}{\prod_{j=1}^n\Gamma(\frac{-a_j+\nu}{2})}\cdot\frac{1}{(4\pi\sqrt{-1})^{n-1}} \\ &\times \int_{z_1,\dots,z_{n-1}}\prod_{j=1}^{n-1}\Gamma\Big(\frac{s_j - z_j}{2}\Big) \\ &\times \prod_{j=1}^{n-1}\Gamma\Big(\frac{s_{j+1} - z_j + \nu}{2}\Big) \\ &\times T_{n,a}(z_1,\dots,z_{n-1})\prod_{j=1}^{n-1}dz_j. \end{aligned} \tag{2.4}$$

Here we understand $s_n = 0$.

To prove this lemma we use Proposition 3 and the following lemma:

Lemma 5. *(a) Barnes' first lemma* [1]

$$\frac{1}{2\pi\sqrt{-1}}\int_z \Gamma(a+z)\Gamma(b+z)\Gamma(c-z)\Gamma(d-z)\,dz$$

$$= \frac{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+b+c+d)}.$$

(b) For complex numbers a_j, b_j, α, β with $a_0 = b_0 = 0$, we have

$$\frac{1}{(2\pi\sqrt{-1})^n} \int_{w_1,\dots,w_n} \times \prod_{i=1}^{n} \Gamma(a_j + w_j)\Gamma(a_{j-1} + \alpha + w_j) \times \prod_{i=1}^{n} \Gamma(b_j - w_j)\Gamma(b_{j-1} + \beta - w_j)\, dw_1 \cdots dw_n$$

$$= \frac{\Gamma(a_1+\beta)\Gamma(b_1+\alpha)\Gamma(\alpha+\beta)\Gamma(a_n+b_n)}{\Gamma(a_n+b_n+\alpha+\beta)} \times \frac{1}{(2\pi\sqrt{-1})^{n-1}} \int_{w_1,\dots,w_{n-1}} \times \prod_{j=1}^{n-1} \Gamma(a_j + w_j)\Gamma(a_{j+1} + \beta + w_j) \times \prod_{j=1}^{n-1} \Gamma(b_j - w_j)\Gamma(b_{j+1} + \alpha - w_j) \times dw_1 \cdots dw_{n-1}.$$

Note that of (b) is a consequence of (a).

Now we rewrite the local zeta more explicitly. If $m < n$, the right $O(n)$-invariance of the class one Whittaker functions implies that

$$\Psi(s, W_{n,a}, W_{m,a'}) = 2^m \int_{\mathbb{R}_+^m} W_{n,a}^0(y_1, \dots, y_m, 1, \dots, 1) \times W_{m,a'}^0(y_1, \dots, y_{m-1}) \prod_{j=1}^{m} y_j^{js} \prod_{j=1}^{m} \frac{dy_j}{y_j}.$$

Then from the Mellin inversion we have

$$\Psi(s, W_{n,a}, W_{m,a'}) = \frac{2^{2-n}}{(2\pi\sqrt{-1})^{n-2}} \int_{\substack{s_1,\dots,s_{m-1}\\ s_{m+1},\dots,s_{n-1}}} \times T_{n,a}(s - s_1, 2s - s_2, \dots, (m-1)s - s_{m-1}, ms, s_{m+1}, \dots, s_{n-1}) \times T_{m,a'}(s_1, \dots, s_{m-1}) \cdot \pi^{-\frac{m(m+1)s}{2} - \quad {}_{i=m+1}^{n-1} s_i} \times ds_1 \cdots ds_{m-1} ds_{m+1} \cdots ds_{n-1}.$$

In the case of $m = n$, by multiplying the 'normalizing factor' of Eisenstein series the target integral is

$$\Psi(s, W_{n,a}, W_{n,a'}) = \frac{\Gamma_{\mathbb{R}}(2ns)}{(4\pi\sqrt{-1})^{n-1}} \int_{s_1,\dots,s_{n-1}} \times T_{n,a}(s - s_1, 2s - s_2, \dots, (n-1)s - s_{n-1}) \times T_{n,a'}(s_1, \dots, s_{n-1})\, \pi^{-\frac{n(n+1)s}{2}} ds_1 \cdots ds_{n-1}.$$

3. The case of $m = n, n-1$

Stade proved the coincidence of the archimedean zeta integrals and the L-factors in the case of $m = n, n-1$.

Theorem 6. [7], [8] *When $m = n-1$ and $m = n$, we have*

$$\frac{\Psi(s, W_{n,a}, W_{m,a'})}{L(s, \pi_v, \pi_{v'})} = \frac{\widetilde{\Psi}(1-s, \widetilde{W}_{n,a}, \widetilde{W}_{m,a'})}{L(1-s, \widetilde{\pi}_v, \widetilde{\pi}_{v'})} = 1.$$

Remark 1. In his proof of Theorem 6, Stade used the recursive relation between $T_{n,a}$ and $T_{n-2,a}$ given in [7]. We can simplify it by using Propositions 3 and 4(a). Actually, the computation of $\Psi_v(s, W_{n,a}, W_{n,a'})$ and $\Psi_v(s, W_{n,a}, W_{n-1,a'})$ are reduced to the evaluation of $\Psi(s, W_{n,a}, W_{n-1,a'})$ and $\Psi(s, W_{n-1,a}, W_{n-1,a'})$, respectively. Thus our result comes from the case of $GL_2 \times GL_2$, that is, Barnes' first lemma.

4. The case of $m = n-2$

Our result is the following:

Theorem 7. *When $m = n-2$, we have*

$$\frac{\Psi(s, W_{n,a}, W_{m,a'})}{L(s, \pi_v, \pi_{v'})} = \frac{\widetilde{\Psi}(1-s, \widetilde{W}_{n,a}, \widetilde{W}_{m,a'})}{L(1-s, \widetilde{\pi}_v, \widetilde{\pi}_{v'})} = \frac{1}{4\pi\sqrt{-1}} \int_w \frac{\prod_{j=1}^{n} \Gamma_{\mathbb{R}}(w - a_j)}{\prod_{j=1}^{n-2} \Gamma_{\mathbb{R}}(s + w + a'_j)} dw.$$

Remark 2. The case of $(m, n) = (3, 1)$ is discussed in [3].

4.1. The computation of $\Psi(s, W_{n,a}, W_{n-2,a'})$.

We use Proposition 3 for $T_{n,a}$ with $s_j \to js - s_j$ $(1 \le j \le n-3)$, $s_{n-2} \to (n-2)s$ and $s_{n-1} \to s_{n-1}$, we get

$$\Psi(s, W_{n,a}, W_{n-2,a'}) = \frac{\pi^{-\frac{(n-2)(n-1)s}{2}}}{(4\pi\sqrt{-1})^{2n-4}} \int_{\substack{s_1,\dots,s_{n-3},s_{n-1}\\ z_1,\dots,z_{n-2}}} \times \prod_{j=1}^{n-3} \Gamma\Big(\frac{j(s - \frac{a_1}{n-1}) - s_j - z_{j-1} + \frac{na_1}{n-1}}{2}\Big) \times \prod_{j=1}^{n-3} \Gamma\Big(\frac{j(s - \frac{a_1}{n-1}) - s_j - z_j}{2}\Big)$$

$$\times \Gamma\Big(\frac{(n-2)s - z_{n-3}}{2} + \frac{a_1}{n-1}\Big) \times \Gamma\Big(\frac{(n-2)s - z_{n-2}}{2} - \frac{(n-2)a_1}{2(n-1)}\Big) \times \Gamma\Big(\frac{s_{n-1} - z_{n-2}}{2} + \frac{a_1}{2(n-1)}\Big)\Gamma\Big(\frac{s_{n-1} - a_1}{2}\Big) \times T_{n-1,b}(z_1, \dots, z_{n-2})\, T_{n-2,a'}(s_1, \dots, s_{n-3}) \times \pi^{-s_{n-1}} dz_1 \cdots dz_{n-2}\, ds_1 \cdots ds_{n-3}\, ds_{n-1}.$$

We apply Proposition 4 (a) with $\nu = s + a_1$ to perform $ds_1 \cdots ds_{n-3}$. Then we have

$$\Psi(s, W_{n,a}, W_{n-2,a'}) = \prod_{j=1}^{n-2} \Gamma\Big(\frac{s + a_1 + a'_j}{2}\Big)$$
$$\times \frac{\pi^{-\frac{(n-2)(n-1)s}{2}}}{(4\pi\sqrt{-1})^{n-1}} \int_{s_{n-1}, z_1, \ldots, z_{n-2}}$$
$$\times \Gamma\Big(\frac{(n-2)s - z_{n-2}}{2} - \frac{(n-2)a_1}{2(n-1)}\Big)$$
$$\times \Gamma\Big(\frac{s_{n-1} - z_{n-2}}{2} + \frac{a_1}{2(n-1)}\Big)\Gamma\Big(\frac{s_{n-1} - a_1}{2}\Big)$$
$$\times T_{n-2,a'}\Big(s - \frac{a_1}{n-1} - z_1, \ldots,$$
$$\ldots, (n-3)(s - \frac{a_1}{n-1}) - z_{n-3}\Big)$$
$$\times T_{n-1,b}(z_1, \ldots, z_{n-2}) \pi^{-s_{n-1}}\, ds_{n-1}\, dz_1 \cdots dz_{n-2}.$$

Now we apply Proposition 3 for $T_{n-1,b}$ and integrate with respect to $z_1, \ldots, z_{n-3}$ by Proposition 4 (a) and z_{n-2} by Barnes' first lemma to reach

$$\Psi(s, W_{n,a}, W_{n-2,a'})$$
$$= \prod_{j=1}^{n-2} \Gamma\Big(\frac{s + a_1 + a'_j}{2}\Big)\Gamma\Big(\frac{s + a_2 + a'_j}{2}\Big)$$
$$\times \frac{\pi^{-\frac{(n-2)(n-1)s}{2}}}{(4\pi\sqrt{-1})^{n-2}} \int_{s_{n-1}, w_1, \ldots, w_{n-3}}$$
$$\times \Gamma\Big(\frac{s_{n-1} - a_1}{2}\Big)\Gamma\Big(\frac{s_{n-1} - a_2}{2}\Big)$$
$$\times \frac{\Gamma(\frac{(n-2)s'}{2})\Gamma(\frac{s'_{n-1} - w_{n-3}}{2})}{\Gamma(\frac{(n-2)s' + s'_{n-1} - w_{n-3}}{2})}$$
$$\times T_{n-2,a'}(s' - w_1, \ldots, (n-3)s' - w_{n-3})$$
$$\times T_{n-2,c}(w_1, \ldots, w_{n-3})$$
$$\times \pi^{-s_{n-1}}\, ds_{n-1} dw_1 \ldots dw_{n-3},$$

where

$$s' = s - \frac{a_1 + a_2}{n-2}, \quad s'_{n-1} = s_{n-1} + \frac{a_1 + a_2}{n-2}$$

and $c = (c_1, \ldots, c_{n-2})$ with

$$c_j = b_{j+1} + \frac{b_1}{n-2} = a_{j+2} + \frac{a_1 + a_2}{n-2}.$$

Now we use Proposition 4 (a) for $T_{n-2,a'}$ with $\nu = s' + s'_{n-1}$ and $s_j = js' - w_j$, and apply Proposition 3 for $T_{n-2,c}$. Then we get

$$\Psi(s, W_{n,a}, W_{n-2,a'})$$
$$= \prod_{j=1}^{n-2} \Gamma\Big(\frac{s + a_1 + a'_j}{2}\Big)\Gamma\Big(\frac{s + a_2 + a'_j}{2}\Big)$$
$$\times \frac{\pi^{-\frac{(n-2)(n-1)s}{2}}}{(4\pi\sqrt{-1})^{3n-9}} \int_{\substack{s_{n-1}, w_1, \ldots, w_{n-3}, \\ p_1, \ldots, p_{n-3}, q_1, \ldots, q_{n-4}}}$$
$$\times \frac{\Gamma(\frac{s_{n-1} - a_1}{2})\Gamma(\frac{s_{n-1} - a_2}{2})\Gamma(\frac{(n-2)s'}{2})}{\prod_{j=1}^{n-2} \Gamma(\frac{s + s_{n-1} + a'_j}{2})}$$
$$\times \Gamma\Big(\frac{s'_{n-1} - w_{n-3}}{2}\Big) \prod_{j=1}^{n-3} \Gamma\Big(\frac{js' - p_j - w_j}{2}\Big)$$
$$\times \prod_{j=1}^{n-3} \Gamma\Big(\frac{js' - p_j - w_{j-1} + s'_{n-1}}{2}\Big)$$
$$\times \prod_{j=1}^{n-3} \Gamma\Big(\frac{w_j - q_{j-1}}{2} + \frac{(n-j-2)c_1}{2(n-3)}\Big)$$
$$\times \Gamma\Big(\frac{w_j - q_j}{2} - \frac{jc_1}{2(n-3)}\Big)$$
$$\times T_{n-2,a'}(p_1, \ldots, p_{n-3})\, T_{n-3,d}(q_1, \ldots, q_{n-4})$$
$$\times \pi^{-s_{n-1}} ds_{n-1} dw_1 \cdots dw_{n-3}$$
$$\times dp_1 \cdots dp_{n-3} dq_1 \cdots dq_{n-4},$$

where $d = (d_1, \ldots, d_{n-3})$ with $d_k = c_{k+1} + \frac{c_1}{n-3}$.

By using Lemma 5 (b), we rearrange the integration with respect to $w_1, \ldots, w_{n-3}$ to arrive at

$$\Psi(s, W_{n,a}, W_{n-2,a'})$$
$$= \prod_{j=1}^{n-2} \Gamma\Big(\frac{s + a_1 + a'_j}{2}\Big)\Gamma\Big(\frac{s + a_2 + a'_j}{2}\Big)$$
$$\times \frac{\pi^{-\frac{(n-2)(n-1)s}{2}}}{(4\pi\sqrt{-1})^{3n-10}} \int_{\substack{s_{n-1}, w_1, \ldots, w_{n-4}, \\ p_1, \ldots, p_{n-3}, q_1, \ldots, q_{n-4}}}$$
$$\times \frac{\Gamma(\frac{s_{n-1} - a_1}{2})\Gamma(\frac{s_{n-1} - a_2}{2})\Gamma(\frac{s_{n-1} - a_3}{2})\Gamma(\frac{(n-2)s'}{2})}{\prod_{j=1}^{n-2} \Gamma(\frac{s + s_{n-1} + a'_j}{2})}$$
$$\times \Gamma\Big(\frac{s' - p_1 + c_1}{2}\Big) \prod_{j=1}^{n-4} \Gamma\Big(\frac{(j+1)s' - p_{j+1} - w_j}{2}\Big)$$
$$\times \prod_{j=1}^{n-3} \Gamma\Big(\frac{js' - p_j - w_j - c_1}{2}\Big)$$
$$\times \prod_{j=1}^{n-4} \Gamma\Big(\frac{-q_j + w_j}{2} + \frac{(n-3-j)}{2(n-3)} c_1\Big)$$
$$\times \prod_{j=1}^{n-3} \Gamma\Big(\frac{s'_{n-1} - q_{j-1} + w_j}{2} + \frac{(n-2-j)}{2(n-3)} c_1\Big)$$
$$\times T_{n-2,a'}(p_1, \ldots, p_{n-3})\, T_{n-3,d}(q_1, \ldots, q_{n-4})$$
$$\times \pi^{-s_{n-1}} ds_{n-1} dw_1 \cdots dw_{n-4}$$
$$\times dp_1 \cdots dp_{n-3} dq_1 \cdots dq_{n-4},$$

Now we use Proposition 4 (a) and (b) for the integrations p_j and q_j, respectively we can find that

$$\Psi(s, W_{n,a}, W_{n-2,a'})$$
$$= \prod_{j=1}^{n-2} \prod_{i=1}^{3} \Gamma\Big(\frac{s + a_i + a'_j}{2}\Big)$$

$$\times \frac{\pi^{-\frac{(n-2)(n-1)s}{2}}}{(4\pi\sqrt{-1})^{n-3}} \int_{\substack{s_{n-1} \\ w_1,\dots,w_{n-4}}} \frac{\prod_{j=1}^{n} \Gamma(\frac{s_{n-1}-a_j}{2})}{\prod_{j=1}^{n-2} \Gamma(\frac{s+s_{n-1}+a'_j}{2})}$$
$$\times T_{n-2,a'}(s''-w_1,\dots,(n-4)s''-w_{n-4}, (n-3)s'')$$
$$\times T_{n-3,d}(w_1,\dots,w_{n-4})$$
$$\times \pi^{-s_{n-1}} ds_{n-1} dw_1 \cdots dw_{n-4}$$

with $s'' = s' - \frac{c_1}{n-3}$.

The integration with respect to w_j is nothing but $\Psi(s'', W_{n-2,a'}, W_{n-3,d})$. Thus Theorem 6 implies that

$$\Psi(s, W_{n,a}, W_{n-2,a'}) = \prod_{i=1}^{n} \prod_{j=1}^{n-2} \Gamma_{\mathbb{R}}(s + a_i + a'_j) \times \frac{1}{4\pi\sqrt{-1}} \int_{s_{n-1}} \frac{\prod_{j=1}^{n} \Gamma_{\mathbb{R}}(s_{n-1} - a_j)}{\prod_{j=1}^{n-2} \Gamma_{\mathbb{R}}(s + s_{n-1} + a'_j)} ds_{n-1},$$

as desired.

4.2. **The computation of $\widetilde{\Psi}(s, W_{n,a}, W_{n-2,a'})$.** By the right K-invariance of Whittaker functions, our target is the integral

$$\widetilde{\Psi}(s, W_{n,a}, W_{n-2,a'}) = \int_{\mathbb{R}_+^{n-2}} \int_{\mathbb{R}^{n-2}} \times W_{n,a}\begin{pmatrix} \bar{y} & & \\ x & 1 & \\ & & 1 \end{pmatrix} W_{n-2,a'}(\bar{y}) \times \prod_{k=1}^{n-2} y_k^{k(s-n+k+1)} \prod_{k=1}^{n-2} dx_k \prod_{k=1}^{n-1} \frac{dy_k}{y_k},$$

with $\bar{y} = \mathrm{diag}(y_1 \cdots y_{n-2}, y_2 \cdots y_{n-2}, \dots, y_{n-2})$, $x = (x_1, \dots, x_{n-2})$. For $y_{n-1} \in \mathbb{C}$, we define the integral $\widetilde{\Psi}(s, W_{n,a}, W_{n-2,a'}; y_{n-1})$ by replacing

$$\begin{pmatrix} \bar{y} & & \\ x & 1 & \\ & & 1 \end{pmatrix} \to \begin{pmatrix} y_{n-1}\bar{y} & & \\ x & y_{n-1} & \\ & & 1 \end{pmatrix}$$

in $\widetilde{\Psi}(s, W_{n,a}, W_{n-2,a'})$. We consider the Mellin transform

$$\widetilde{\Psi}(s, w) := \int_0^\infty \widetilde{\Psi}(s, W_{n,a}, W_{n-2,a'}; y_{n-1}) \times y_{n-1}^w \frac{dy_{n-1}}{y_{n-1}}.$$

Then

$$\widetilde{\Psi}(s, w) = 2^{n-2} \int_{\mathbb{R}_+^{n-1}} \int_{\mathbb{R}^{n-2}} \times W_{n,a}\big(X(x, y)\big) W_{n-2,a'}(y_1, \dots, y_{n-3}) \times \prod_{k=1}^{n-2} y_k^{k(s-n+k+1)} \cdot y_{n-1}^w \prod_{k=1}^{n-2} dx_k \prod_{k=1}^{n-1} \frac{dy_k}{y_k}.$$

Here we denote by

$$X(x, y) \equiv X((x_1, \dots, x_{n-2}), (y_1, \dots, y_{n-1})) = \begin{pmatrix} y_{n-1}\bar{y} & & \\ x & y_{n-1} & \\ & & 1 \end{pmatrix}.$$

Lemma 8. *Let $\bar{x} := X((x_1, \dots, x_{n-2}), (1, \dots, 1))$. Then we have the Iwasawa decomposition $\bar{x} = n(\bar{x})a(\bar{x})k(\bar{x})$ $(n(\bar{x}) \in N_n(\mathbb{R}), a(\bar{x}) \in A, k(\bar{x}) \in O(n))$ can be written as*

$$n(\bar{x})_{ij} = \begin{cases} -\frac{x_i x_j}{p_j} & \text{if } 1 \le i < j \le n-1, \\ \frac{x_i}{p_{n-1}} & \text{if } j = n, \end{cases}$$
$$a(\bar{x})_{ii} = \sqrt{\frac{p_i}{p_{i+1}}}, \quad 1 \le i \le n-1,$$

where $p_1 = p_n = 1$ and $p_i = 1 + x_1^2 + \cdots + x_{i-1}^2$ for $2 \le i \le n-1$.

In view of

$$X(x, y) = X\big((0, \dots, 0), (y_1, \dots, y_{n-1})\big) \times X\big((\frac{x_1}{y_{n-1}}, \dots, \frac{x_{n-2}}{y_{n-1}}), (1, \dots, 1)\big)$$

and the lemma above, the substitution

$$x_k \to -x_k y_{n-1} \quad (1 \le k \le n-2)$$

implies that

$$\widetilde{\Psi}(s, w) = 2^{n-1} \int_{\mathbb{R}_+^{n-1}} \int_{\mathbb{R}^{n-2}} \times W_{n,a}\Big(y_1 \frac{\sqrt{p_1 p_3}}{p_2}, \dots, y_{n-2} \frac{\sqrt{p_{n-2} p_n}}{p_{n-1}}, y_{n-1}\sqrt{p_{n-1}}\Big)$$
$$\times \exp\Big\{-2\pi\sqrt{-1}\Big(\sum_{k=1}^{n-3} \frac{x_k x_{k+1}}{p_{k+1}} y_k + \frac{x_{n-2}}{p_{n-1}} y_{n-2}\Big)\Big\}$$
$$\times W_{n-2,a'}(y_1, \dots, y_{n-3}) \times \prod_{k=1}^{n-2} y_k^{k(s-n+k+1)} \cdot y_{n-1}^{w+n-2} \prod_{k=1}^{n-2} dx_k \prod_{k=1}^{n-1} \frac{dy_k}{y_k}.$$

Next we replace

$$y_k \to \frac{p_{k+1}}{\sqrt{p_k p_{k+2}}} y_k \quad (1 \le k \le n-2), \qquad y_{n-1} \to \frac{y_{n-1}}{\sqrt{p_{n-1}}}.$$

Then we have

$$\widetilde{\Psi}(s, w) = 2^{n-1} \int_{\mathbb{R}_+^{n-1}} \int_{\mathbb{R}^{n-2}} \times W_{n,a}(y_1, \dots, y_{n-1}) \times W_{n-2,a'}\Big(y_1 \frac{p_2}{\sqrt{p_1 p_3}}, \dots, y_{n-3} \frac{p_{n-2}}{\sqrt{p_{n-3} p_{n-1}}}\Big)$$
$$\times \exp\Big\{-2\pi\sqrt{-1}\Big(\sum_{k=1}^{n-3} \frac{x_k x_{k+1}}{p_k p_{k+2}} y_k + \frac{x_{n-2}}{p_{n-2}} y_{n-2}\Big)\Big\}$$

$$\times \prod_{k=1}^{n-2}\Big(\frac{p_{k+1}}{\sqrt{p_kp_{k+2}}}y_k\Big)^{k(s-n+k+1)}\Big(\frac{y_{n-1}}{\sqrt{p_{n-1}}}\Big)^{w+n-2}$$
$$\times \prod_{k=1}^{n-2} dx_k \prod_{k=1}^{n-1}\frac{dy_k}{y_k}.$$

In view of $W_{n,a}(y)=y^{\rho}W^0_{n,a}(y)$, we get

$$\widetilde{\Psi}(s,w)=2^{n-1}\int_{\mathbb{R}_+^{n-1}}\int_{\mathbb{R}^{n-2}}$$
$$\times W^0_{n,a}(y_1,\ldots,y_{n-1})$$
$$\times W^0_{n-2,a'}\Big(y_1\frac{p_2}{\sqrt{p_1p_3}},\ldots,y_{n-3}\frac{p_{n-2}}{\sqrt{p_{n-3}p_{n-1}}}\Big)$$
$$\times \exp\Big\{-2\pi\sqrt{-1}\Big(\sum_{k=1}^{n-3}\frac{x_kx_{k+1}}{p_kp_{k+2}}y_k+\frac{x_{n-2}}{p_{n-2}}y_{n-2}\Big)\Big\}$$
$$\times \prod_{k=1}^{n-2} y_k^{ks}\cdot y_{n-1}^{w+\frac{3n-5}{2}}\prod_{k=2}^{n-2}\frac{1}{\sqrt{p_k}}\cdot p_{n-1}^{\frac{(n-1)(2s-3)}{4}-\frac{w}{2}}$$
$$\times \prod_{k=1}^{n-2} dx_k\prod_{k=1}^{n-1}\frac{dy_k}{y_k}.$$

Now we change the variables $(x_1,\ldots,x_{n-2})\to(x'_1,\ldots,x'_{n-2})$ by

$$x_k=x'_k\sqrt{\frac{p'_{n-1}}{p'_{n-k-1}p'_{n-k}}}\quad(1\le k\le n-2),$$

where $p'_1=1$ and $p'_k=1+(x'_{n-k})^2+\cdots+(x'_{n-2})^2$ for $2\le k\le n-1$. This implies

$$x'_k=x_k\sqrt{\frac{p_{n-1}}{p_kp_{k+1}}};\quad p_k=\frac{p'_{n-1}}{p'_{n-k}};$$
$$\prod_{k=1}^{n-2}\frac{dx_k}{\sqrt{p_k}}=\prod_{k=1}^{n-2}\frac{dx'_k}{\sqrt{p'_k}}.$$

Then we arrive at

$$\widetilde{\Psi}(s,w)=2^{n-1}\int_{\mathbb{R}_+^{n-1}}W^0_{n,a}(y_1,\ldots,y_{n-1})$$
$$\int_{\mathbb{R}^{n-2}}W^0_{n-2,a'}\Big(y_1\frac{\sqrt{p'_{n-1}p'_{n-3}}}{p'_{n-2}},\ldots,y_{n-3}\frac{\sqrt{p'_3p'_1}}{p'_2}\Big)$$
$$\times \exp\Big\{-2\pi\sqrt{-1}\Big(\sum_{k=1}^{n-3}\frac{x'_kx'_{k+1}}{p'_{n-k-1}}y_k+x'_{n-2}y_{n-2}\Big)\Big\}$$
$$\times p'^{\,\frac{(n-1)(2s-3)}{4}-\frac{w}{2}}_{n-1}\prod_{k=1}^{n-2}\frac{dx'_k}{\sqrt{p'_k}}$$
$$\times \prod_{k=1}^{n-2}y_k^{ks}\cdot y_{n-1}^{w+\frac{3n-5}{2}}\prod_{k=1}^{n-1}\frac{dy_k}{y_k}.$$

In view of Lemma 2, the integration with respect to x'_k gives Whittaker functions on $GL(n-1,\mathbb{R})$:

$$\int_{\mathbb{R}^{n-2}}\prod_{k=1}^{n-2}dx'_k=\prod_{k=1}^{n-2}\frac{1}{\Gamma_{\mathbb{R}}(1+\nu_k-\nu_{n-1})}$$
$$\times\prod_{k=1}^{n-2}y_{n-k-1}^{-\frac{n-k-1}{n-2}\nu_{n-1}}$$
$$\times W^0_{n-1,(\nu_1,\ldots,\nu_{n-1})}(y_{n-2},\ldots,y_1),$$

with

$$\nu_k=-a'_k-(s-\tfrac{3}{2})+\tfrac{w-1}{n-1}\quad(1\le k\le n-2),$$
$$\nu_{n-1}=(n-2)(s-\tfrac{3}{2})-\tfrac{n-2}{n-1}(w-1).$$

Then we reach

$$\widetilde{\Psi}(s,w)=2^{n-1}\prod_{k=1}^{n-2}\frac{1}{\Gamma_{\mathbb{R}}(1+\nu_k-\nu_{n-1})}$$
$$\times\int_{\mathbb{R}_+^{n-1}}W^0_{n,a}(y_1,\ldots,y_{n-1})$$
$$\times W^0_{n-1,(-\nu_1,\ldots,-\nu_{n-1})}(y_1,\ldots,y_{n-2})$$
$$\times\prod_{k=1}^{n-1}y_k^{k(\frac{3}{2}+\frac{w-1}{n-1})}\prod_{k=1}^{n-1}\frac{dy_k}{y_k}.$$

This is the archimedean zeta integral for $GL_n\times GL_{n-1}$ and thus Theorem 6 leads to

$$\widetilde{\Psi}(s,w)=\prod_{j=1}^{n}\prod_{k=1}^{n-2}\Gamma_{\mathbb{R}}(s+a_j+a'_k)$$
$$\times\frac{\prod_{j=1}^{n}\Gamma_{\mathbb{R}}(w'+a_j)}{\prod_{k=1}^{n-2}\Gamma_{\mathbb{R}}(w'+1-s-a'_k)}$$

with $w'=w-(n-2)(s-\frac{3}{2})+\frac{1}{2}$. Thus Mellin inversion implies that

$$\widetilde{\Psi}(s,W_{n,a},W_{n-2,a'})=\prod_{j=1}^{n}\prod_{k=1}^{n-2}\Gamma_{\mathbb{R}}(s+a_j+a'_k)$$
$$\times\frac{1}{4\pi\sqrt{-1}}\int_w\frac{\prod_{j=1}^{n}\Gamma_{\mathbb{R}}(w+a_j)}{\prod_{k=1}^{n-2}\Gamma_{\mathbb{R}}(w+1-s-a'_k)}\,dw.$$

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